STABLE LOCAL DYNAMICS: EXPANSION, QUASI-CONFORMALITY AND ERGODICITY

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Abstract. In this paper we study stable ergodicity of the action of groups of diffeomorphisms on smooth manifolds. The existence of such actions is known only on one dimensional manifolds. The aim of this paper is to introduce a geometric method to overcome this restriction and for constructing higher dimensional examples. In particular, we show that every closed manifold admits stably ergodic finitely generated group actions by diffeomorphisms of class $\mathcal{C}^{1+\alpha}$. We also prove the stable ergodicity of certain algebraic actions including the natural action of a generic pair of matrices near the identity on a sphere of arbitrary dimension. These are consequences of the *quasi-conformal blender*, a local and stable mechanism/phenomenon introduced in this paper, which encapsulates our method to prove stable local ergodicity by providing quasi-conformal orbits with fine controlled geometry. The quasi-conformal blender is developed in the context of pseudo-semigroup actions of locally defined smooth diffeomorphisms. This allows for applications in different settings, including for the smooth foliations of arbitrary codimension.

Contents

1. Introduction

Let G be a subgroup of $\mathrm{Diff}^{1+\alpha}(M)$, the space of all diffeomorphisms with α-Hölder derivative on a smooth Riemannian manifold M endowed with the $\mathcal{C}^1$ topology. The action of G is *minimal* if every orbit is dense. Also, the action of G is *ergodic (w.r.t. Leb.)* if every G-invariant set of positive Lebesgue measure in M has full measure. Recall that for a map f, a measurable set $S \subseteq M$ is called f*-invariant* if $f(S) \subseteq S$ up to a set of zero Lebesgue measure. This definition of ergodicity concerns only the class of the Lebesgue measure, and the invariance of the Lebesgue measure is not assumed. A direct consequence of ergodicity is that Lebesgue almost every point has a dense orbit. For $\mathcal{F} \subseteq \mathrm{Diff}^{1+\alpha}(M)$, we say that the action of $\langle \mathcal{F} \rangle$, the group generated by $\mathcal{F}$, is *stably ergodic* if the action of the group generated by any $\mathcal{C}^1$ small perturbation of $\mathcal{F}$ in $\mathrm{Diff}^{1+\alpha}(M)$ is ergodic.

A remarkable result regarding ergodicity with respect to the Lebesgue measure (in the absence of an invariant measure in its class), proved by Katok and Herman, asserts that every $\mathcal{C}^2$ minimal circle diffeomorphism is ergodic (cf. [Her79, KH95, Nav11]). Note that the ergodicity in this result is not stable under perturbation. The following theorem of Sullivan provides examples of stably ergodic actions by circle diffeomorphisms. Indeed, the existence of stably ergodic actions by diffeomorphisms is known only in dimension one and the aim of this paper is to introduce a mechanism for stable ergodicity in every dimension.

Theorem 1.1 (Sullivan)**.** *Let G be a group of $\mathcal{C}^{1+\alpha}$ circle diffeomorphisms, with $\alpha > 0$. Assume that for all $x \in \mathbb{S}^1$, there exists some $g \in G$ such that $g'(x) > 1$. If the action of G is minimal, then it is ergodic.*

From the proof of this theorem, one can deduce that the action of the groups generated by certain finite subsets of G are indeed stably ergodic (cf. [Nav04, SS85]). This for instance implies that the group generated by an irrational rotation and a Morse-Smale diffeomorphism is an example of a stably ergodic action by diffeomorphisms in $\mathrm{Diff}^{1+\alpha}(\mathbb{S}^1)$.

In dimension one, every action is conformal, i.e. balls are mapped to balls. This is a crucial fact in the study of group actions on a one dimensional manifold and particularly in Theorem 1.1 above (cf. [DKN18, Nav18] and their references for recent developments). The idea of generalizing Sullivan's proof to higher dimensions is, of course, not new. For instance, generalizations of Theorem 1.1 are proved for conformal actions in higher dimensions (cf. [DK07a, BFMS17]). However, such generalizations do not provide stably ergodic actions, since conformality (or quasi-conformality) of an action is not stable in higher dimensions.

In this paper, we develop a geometric approach generalizing Sullivan's method to higher dimensions. It allows us to control the geometry of images of small open balls under certain orbit-branches, in a very stable way. In particular, it yields the following variant of Theorem 1.1. Such controlled geometry along the orbits is a crucial part of our proofs and may be of its own independent interest.

Theorem A. *Let G be a group of $\mathcal{C}^{1+\alpha}$ diffeomorphisms on a smooth closed Riemannian manifold M, with $\alpha > 0$. Assume that for any $(x, v) \in T^1M$, there exists some $g \in G$ such that $m(D_xg) > 1$ and $\|\hat{D}_xg|_{v^\perp}\| < 1$. If the action of G is minimal, then it is ergodic. Moreover, G contains a finite subset generating a stably ergodic group action.*

Here, $\hat{D}_x(f) := \sigma^{-1/d} D_xf$ denotes the normalized derivative, where $\sigma > 0$ is the Jacobian of f at x. Also, $m(.)$ is the co-norm of a linear map, and $v^\perp$ is the linear subspace orthogonal to a vector v. If $\dim(M) = 1$, the statement of this theorem is exactly the one of Theorem 1.1, since $\|\hat{D}_xg|_{v^\perp}\| \equiv 0$ on T^1M. The conclusion of Theorem A is not sensitive to the choice of Riemannian metric and it suffices to ensure its assumptions for a metric on M.

Theorem A and its counterpart for the pseudo-groups of locally defined diffeomorphisms (Theorem 6.4) are obtained from a local and stable mechanism for ergodicity introduced in Theorem F. It also allows us to prove the following.

Theorem B. *Every closed manifold M admits a stably ergodic semigroup action generated by two diffeomorphisms in* $\mathrm{Diff}^{1+\alpha}(M)$.

This theorem gives the first example of a stably ergodic finitely generated group action in $\mathrm{Diff}^{1+\alpha}(M)$ on a manifold M of dimension greater than one. The notion of stable ergodicity in the space $\mathrm{Diff}^{1+\alpha}(M)$ should not be mistaken with the notion of stable ergodicity within the class of volume-preserving diffeomorphisms $\mathrm{Diff}^{1+\alpha}_{\mathrm{vol}}(M)$. In fact, on a closed surface S, we observe that the action of any cyclic subgroup of $\mathrm{Diff}^{1+\alpha}(S)$ is not stably ergodic, while area-preserving Anosov diffeomorphisms of $\mathbb{T}^2$ are stably ergodic in $\mathrm{Diff}^{1+\alpha}_{\mathrm{vol}}(\mathbb{T}^2)$. Such observations indicate that the number of generators in Theorem B is optimal (See Section 7 for further discussion).

On the other hand, $\mathcal{C}^{1+\alpha}$ expanding endomorphisms are stably ergodic. One way to show it is by considering a dynamically defined sequence of partitions with arbitrarily small diameters, such that every element of each partition is eventually mapped to a ball of uniform size. An alternative approach in this setting is based on the functional analytic methods to show the existence of a unique ergodic invariant measure in the class of Lebesgue measure (cf. [VO16, Rue04, Krz78, Sac74]). While this approach has been extended to various settings including maps with non-uniform expansions and singularities (cf. [ABV00, VV10]), it requires further developments in the setting of (pseudo) group actions.

We should also mention the striking results on ergodic theory of groups of surface diffeomorphisms in [BR17], which give a classification of stationary measures for smooth group actions in dimension 2 and yield remarkable examples of stably ergodic group actions in the space of area-preserving surface diffeomorphisms [Liu16, Chu20]. On the other hand, the stable ergodicity of finitely generated dense subgroups of isometries of even dimensional spheres within the class of sufficiently smooth volume-preserving diffeomorphisms is proved in [DK07b]. Unfortunately, the results in these remarkable papers are not enough for showing stable ergodicity beyond the conservative setting (cf. §7.3).

The following algebraic example is a consequence of Theorem A.

Theorem C. *Let $d \geq 1$ and $\mathcal{F}$ be a finite subset of* $\mathrm{SL}(d+1,\mathbb{R})$. *Assume that the closure of $\langle \mathcal{F} \rangle$ strictly contains* $\mathrm{SO}(d+1)$. *Then, the natural action of $\langle \mathcal{F} \rangle$ on $\mathbb{S}^d$ is stably ergodic in* $\mathrm{Diff}^{1+\alpha}(\mathbb{S}^d)$. *Moreover, it is $\mathcal{C}^1$-robustly minimal.*

Here, the action of $A \in \mathrm{SL}(d+1,\mathbb{R})$ on $\mathbb{S}^d$ is defined by $x \mapsto \frac{Ax}{|Ax|}$. Recall that the action of $\langle \mathcal{F} \rangle$ is called $\mathcal{C}^1$-*robustly minimal* if the action of the group generated by every small perturbation of $\mathcal{F}$ in the space of all $\mathcal{C}^1$ diffeomorphisms is minimal.

It is known that for $d \geq 2$, every generic pair of elements near the identity generates a dense subgroup of $\mathrm{SL}(d,\mathbb{R})$ [Kur51].

Corollary D. *The family $\mathcal{F}$ in Theorem C can be a generic pair near the identity element in* $\mathrm{SL}(d+1,\mathbb{R})$.

Recall that by fixing a probability distribution on the acting semigroup, a stationary measure is a measure which remains invariant in average. The main results of this paper imply the following uniqueness result for ergodic stationary measures.

Corollary E. *In the setting of either Theorems A,B or C, if the acting semigroup is finitely generated, for any probability distribution on generators, there is at most* *one absolutely continuous ergodic stationary measure, and it is equivalent to the Lebesgue measure.*

The problem of ergodicity is more subtle for the pseudo-groups of localized dynamics, where the restriction of the maps to a given domain are considered. While the localized dynamics appears in many setting, such as the return maps or the local holonomy of a foliation, it is well-understood only in dimension one. Most basic questions in higher dimensions are open, even for the local affine actions or local homogeneous actions (cf. [BIS17] for a remarkable development for certain algebraic actions). As an application of our main results, one can provide the first examples of foliations of codimension greater than one which are stably ergodic with respect to the transversal Lebesgue measure (cf. [Raj22] for the detailed proof).

1.1. **Quasi-conformal blender.** The next theorem provides a stable and local mechanism for generating quasi-conformal orbits of pseudo-semigroups and to deduce local ergodicity. As mentioned before, it plays a fundamental role in proving our main results stated above. Moreover, it makes the proof of Theorem B constructive, as well as flexible.

We denote $\mathrm{Diff}^{1+}_{\mathrm{loc}}(M) := \bigcup_{\alpha>0} \mathrm{Diff}^{1+\alpha}_{\mathrm{loc}}(M)$, where $\mathrm{Diff}^{s}_{\mathrm{loc}}(M)$ is the space of all $\mathcal{C}^s$ diffeomorphisms $f : U_f \to V_f$ such that U_f and V_f are open subset of M. To obtain the strongest stability results we consider $\mathcal{C}^1$ topology on this space, i.e., two elements of $\mathrm{Diff}^{s}_{\mathrm{loc}}(M)$ are $\mathcal{C}^1$-close if their graphs are $\mathcal{C}^1$-close submanifolds of $M \times M$.

A diffeomorphism $f \in \mathrm{Diff}^{1}_{\mathrm{loc}}(M)$ is called *expanding* if $m(D_x f) > 1$ at every point x in its domain of definition.

Let $\pi : \mathcal{E}(M) \to M$ be the fiber bundle over a Riemannian manifold M of dimension d defined by

$$\mathcal{E}(M) := \big\{(x, \mathbf{v}) : x \in M,\ \mathbf{v} \in (T_x M)^d \text{ and } \det(A_{\mathbf{v}}) = 1\big\},$$

where for $\mathbf{v} := (v_1, \dots, v_d)$, $A_{\mathbf{v}}$ is a $d \times d$ matrix with (i, j)-entries equal to $\langle v_i, v_j \rangle_x$, the inner product of v_i, v_j assigned by the Riemannian metric on $T_x M$. Note that $\mathcal{E}(M)$ is an $\mathrm{SL}^{\pm}(d, \mathbb{R})$ bundle over M, where $\mathrm{SL}^{\pm}(d, \mathbb{R})$ consists of all $d \times d$ matrices with determinant ± 1. Indeed, we consider a metric on $\mathcal{E}(M)$ inducing the following norm on its fibers,

$$\|(x, \mathbf{v})\|_2 := \Big(\sum_{i=1}^{d} \langle v_i, v_i \rangle_x^2\Big)^{\frac{1}{2}}.$$

In particular, whenever M is an open subset of $\mathbb{R}^d$, $\mathcal{E}(M)$ is isomorphic to the trivial bundle $M \times \mathrm{SL}^{\pm}(d, \mathbb{R})$, endowed with the Hilbert-Schmidt norm on the fibers.

For a diffeomorphism $f \in \mathrm{Diff}^{1}_{\mathrm{loc}}(M)$ with $f : U_f \to V_f$, one can naturally define a fiber map $\hat{\mathcal{D}} f : \pi^{-1}(U_f) \to \pi^{-1}(V_f)$ defined by

$$\hat{\mathcal{D}} f(x, \mathbf{v}) := \big(f(x), \hat{D}_x f(\mathbf{v})\big),$$

where $\hat{D}_x f(\mathbf{v}) := (\hat{D}_x f(v_1), \dots, \hat{D}_x f(v_d))$ for $\mathbf{v} = (v_1, \dots, v_d) \in (T_x M)^d$ (cf. Remark 4.3).

Also, we use the notation $f{\downarrow_V} := f|_{V \cap f^{-1}(V)}$ for the restriction of an invertible map f to the set of points in a set V that are mapped to V. Similarly, we denote $\mathcal{F}{\downarrow_V} := \{f{\downarrow_V} : f \in \mathcal{F}\}$ for a family of maps localized to V.

Theorem F (Quasi-conformal blender)**.** *Let $\mathcal{F} \subseteq \mathrm{Diff}^{1+}_{\mathrm{loc}}(M)$ be a family of expanding diffeomorphisms between open subsets of a smooth manifold M. Let $\mathcal{W} \subseteq \mathcal{E}(M)$ be an open set with compact closure, and $V := \pi(\mathcal{W})$. Assume that*

$$\overline{\mathcal{W}} \subseteq \bigcup_{f \in \mathcal{F}} (\hat{\mathcal{D}} f)^{-1}(\mathcal{W}). \tag{1}$$

Then, there exists a real number $r > 0$ such that every measurable $\mathcal{F}{\downarrow_V}$-invariant set S with $\mathrm{Leb}(S \cap V) > 0$ contains a ball of radius r (up to a set of zero measure). Moreover, this property is $\mathcal{C}^1$-stable with uniform $r > 0$.

The principal role of the covering condition (1) is to guarantee the existence of quasi-conformal orbit-branches in the domain V. In dimension one, it is equivalent to $\overline{V} \subseteq \mathcal{F}^{-1}(V)$.

We would like to emphasize on the multiple stability in this theorem. The covering condition (1), the domain V and the radius r are all stable under small perturbations in the $\mathcal{C}^1$ topology, and are independent of the family's regularity class and its corresponding norm. This resembles the idea behind the creation of blenders in partially hyperbolic dynamics. The concept of blender was introduced in the seminal work of Bonatti and Díaz [BD96] as a stable and local mechanism for transitivity. During the last decades, it has been generalized and used in diverse settings (cf. [RRTU11, NP12, ACW17, Ber16] among others).

Most results in this paper are proved for the actions of pseudo-semigroups of locally defined diffeomorphisms. This allows to deal with broader classes of systems. In particular, one can apply Theorem F to get stably ergodic smooth foliations of arbitrary codimension, while the only known examples were of codimension one. This will be discussed in a forthcoming paper. One may expect further applications of the local tool introduced in Theorem F in smooth ergodic theory.

Let say a few words about the proof of Theorem F. The proof of ergodicity is based on the simplest known method, i.e., by means of expansions. Given a set S of positive measure, one has to show that its orbit has full measure. Then, one shows that the iteration of the infinitesimal neighborhood of a Lebesgue density point of S gives large open sets that are mostly contained in the orbit of S. To realize this idea for the locally defined diffeomorphisms (or for the action of diffeomorphisms that each one is expanding at some regions) one needs to control the geometry of balls to remain almost round through the expanding iterations. This requires new techniques which are the main ingredients of this paper. Two main steps are involved. First, we show that covering condition (1) implies that for some $\kappa > 1$, the pseudo-semigroup generated by $\mathcal{F}{\downarrow_V}$ (or its perturbations) has a κ-conformal orbit-branch at every point of V (Theorem 4.1). Second, we obtain a good control of geometry of small balls under iteration of sequences of maps satisfying infinitesimal assumptions of expansion and quasi-conformality (Theorem 3.1). The $\mathcal{C}^{1+\alpha}$ regularity of the sequence is essential in this analysis. These two steps together, roughly, show that diversity of non-conformality may lead to stable and quantitative controlled geometry of certain iterations of small balls. This type of control of geometry, together with the classical distortion control argument allows us to obtain local ergodicity. Further analysis leads to uniform size of radius r, independent of modulus of Hölder regularity of derivatives or their norms.

Organization. The paper is organized as follows. In Section 2, we introduce some definitions, the precise setting, and notations. Section 3 contains a crucial technical step in the proof of the main theorems (Theorem 3.1). Section 4 is devoted to quasi-conformality, where we show that the existence of quasi-conformal orbit-branches is equivalent to condition (1) for some bounded $\mathcal{W}$. Also, two methods are given to verify that condition. They will be used in the proof of Theorems A and B. Section 5 contains the proof of Theorem F and its variants. In Section 6, we prove Theorems A, B and C. In Section 7, some questions related to the main results are discussed.

2. Preliminary definitions and notations

Let M be a boundaryless smooth manifold of dimension d endowed with a Riemannian metric. We denote by $|.|$ the norm induced by this metric on the tangent space. We also denote the measure induced from this metric by Leb., and call it the Lebesgue measure. Furthermore, we denote the ball of radius r with center $x \in M$ by $B(x, r)$. For $k \in \mathbb{N}$ and $\alpha \in (0, 1)$, we say $f : M \to M$ is of class $\mathcal{C}^{k+\alpha}$, whenever f is $\mathcal{C}^k$ and its k-th derivative is α-Hölder continuous.

For a real number $s > 1$, $\mathcal{C}^s$ means $\mathcal{C}^{k+\alpha}$, where $s = k + \alpha$ and k is its integer part. For $f \in \mathrm{Diff}^{1+\alpha}_{\mathrm{loc}}(\mathbb{R}^d)$, the $\mathcal{C}^{1+\alpha}$ norm of f, denoted by $\|f\|_{\mathcal{C}^{1+\alpha}}$ is defined by

$$\|f\|_{\mathcal{C}^{1+\alpha}} := \|f\|_{\mathcal{C}^1} + \sup\Big\{\frac{\|D_x f - D_y f\|}{|x-y|^\alpha} : x, y \in \mathrm{Dom}(f),\ 0 < |x-y| < 1\Big\}.$$

One can use local charts and define $\mathcal{C}^{1+\alpha}$ norm for the smooth maps between open subsets of manifolds.

We use $\mathrm{Diff}^s(M)$ to denote the group of $\mathcal{C}^s$ diffeomorphisms of M. Also, $\mathrm{Diff}^{1+}(M)$ is the union of all $\mathrm{Diff}^s(M)$ with $s > 1$. Throughout the paper, we usually consider the $\mathcal{C}^1$ topology on $\mathrm{Diff}^s(M)$. Denote by $\mathrm{Diff}^s_{\mathrm{loc}}(M)$ the space of all $\mathcal{C}^s$ diffeomorphisms $f : \mathrm{Dom}(f) \to \mathrm{Im}(f)$ such that $\mathrm{Dom}(f)$ and $\mathrm{Im}(f)$ are open subset of M. Two elements of $\mathrm{Diff}^s_{\mathrm{loc}}(M)$ are $\mathcal{C}^l$-close if their graphs are $\mathcal{C}^l$-close submanifolds of $M \times M$, for $l \leq s$. Similarly, $\mathrm{Diff}^{1+}_{\mathrm{loc}}(M)$ is the union of of all $\mathrm{Diff}^s_{\mathrm{loc}}(M)$ with $s > 1$.

For $x \in M$, $U \subseteq M$ and families of maps $\mathcal{F}, \mathcal{G} \subseteq \mathrm{Diff}^1_{\mathrm{loc}}(M)$, we denote $\mathcal{F}(U) := \bigcup_{f \in \mathcal{F}} f(U)$, $\mathcal{F}(x) := \mathcal{F}(\{x\})$, and

$$\mathcal{F} \circ \mathcal{G} := \{f \circ g : f \in \mathcal{F}, g \in \mathcal{G}\}.$$

Also, put $\mathcal{F}^0 = \{\mathrm{Id}\}$ and for $k \in \mathbb{N}$, denote $\mathcal{F}^k := \mathcal{F}^{k-1} \circ \mathcal{F}$. We use $\langle \mathcal{F} \rangle^+$ (resp. $\langle \mathcal{F} \rangle$) for the semigroup (resp. the group) generated by $\mathcal{F}$. By IFS $(\mathcal{F})$, we mean *the iterated function system generated by* $\mathcal{F}$, that is the action of $\langle \mathcal{F} \rangle^+$ on M. Given a finite family $\mathcal{F} = \{f_1, \dots, f_k\} \subseteq \mathrm{Diff}^{1+\alpha}_{\mathrm{loc}}(M)$, the ϵ-neighbourhood of $\mathcal{F}$ in the $\mathcal{C}^1$ topology is the set of all families $\tilde{\mathcal{F}} = \{\tilde{f}_1, \dots, \tilde{f}_k\} \subseteq \mathrm{Diff}^{1+\alpha}_{\mathrm{loc}}(M)$ such that $f_i, \tilde{f}_i$ are ϵ-close in the $\mathcal{C}^1$ topology, for any $i = 1, \dots, k$. Similarly, one can define ϵ-neighbourhood of an infinite family of maps in $\mathrm{Diff}^{1+\alpha}_{\mathrm{loc}}(M)$.

We say a property P holds $\mathcal{C}^1$-*stably* for $\mathcal{F}$ in $\mathrm{Diff}^{1+\alpha}(M)$, if P holds for every $\tilde{\mathcal{F}}$ in a $\mathcal{C}^1$-open neighbourhood of $\mathcal{F}$ in $\mathrm{Diff}^{1+\alpha}(M)$. Also, we say a property P holds $\mathcal{C}^1$-*robustly* for $\mathcal{F}$, if P holds for every $\tilde{\mathcal{F}}$ in a $\mathcal{C}^1$-open neighbourhood of $\mathcal{F}$ in $\mathrm{Diff}^1(M)$. Clearly, by the definition, $\mathcal{C}^1$-robustness is stronger than $\mathcal{C}^1$-stability. Similarly, $\mathcal{C}^1$ stability and robustness are defined for $\mathrm{Diff}^{1+}(M)$, $\mathrm{Diff}^{1+}_{\mathrm{loc}}(M)$ and $\mathrm{Diff}^s_{\mathrm{loc}}(M)$, $s > 1$.

2.1. **Localized dynamics.** For an open set $V \subseteq M$ and $f \in \mathrm{Diff}_{\mathrm{loc}}(M)$ with $f : \mathrm{Dom}(f) \to \mathrm{Im}(f)$, we define the localization of f to V, by $f{\downarrow_V} := f|_{V \cap f^{-1}(V)}$. Clearly, the domain and the image of $f{\downarrow_V}$ are $\mathrm{Dom}(f{\downarrow_V}) = V \cap f^{-1}(V \cap \mathrm{Im}(V))$, and $\mathrm{Im}(f{\downarrow_V}) = f(V \cap \mathrm{Dom}(f)) \cap V$, respectively, and $f{\downarrow_V} : \mathrm{Dom} f{\downarrow_V} \to \mathrm{Im} f{\downarrow_V}$ is a bijective map.

For a family $\mathcal{F}$ of invertible maps, we denote the pseudo-semigroup (resp. pseudo-group) generated by localization of elements of $\mathcal{F}$ to V by $\langle \mathcal{F}{\downarrow_V} \rangle^+$ (resp. $\langle \mathcal{F}{\downarrow_V} \rangle$) and by $\mathrm{IFS}(\mathcal{F}{\downarrow_V})$ the action of this pseudo-semigroup. A *finite orbit-branch of* $\mathrm{IFS}(\mathcal{F}{\downarrow_V})$ *at* x is a sequence $\{x_i\}_{i=0}^n$ in V such that $x_0 = x$ and for any $1 \le i \le n$, there exists $f_i \in \mathcal{F}$ with $x_{i-1} \in \mathrm{Dom}(f_i{\downarrow_V})$ and $f_i(x_{i-1}) = x_i$. Infinite orbit-branches are defined similarly. The orbit of $\mathrm{IFS}(\mathcal{F}{\downarrow_V})$ at $x \in V$, denoted by $\langle \mathcal{F}{\downarrow_V} \rangle^+(x)$, is the set of all points in finite orbit-branches at x. For $S \subseteq V$, define $\langle \mathcal{F}{\downarrow_V} \rangle^+(S) := \cup_{x \in S} \langle \mathcal{F}{\downarrow_V} \rangle^+(x)$.

In this paper, we deal with two basic dynamical concepts, namely, minimality and ergodicity. $\mathrm{IFS}(\mathcal{F}{\downarrow_V})$ is called *minimal* if for any $x \in V$, $\langle \mathcal{F}{\downarrow_V} \rangle^+(x)$ is dense in V. Fixing a measure μ on V, we say a measurable map $f{\downarrow_V}$ is *non-singular* with respect to μ, if $f_*\big(\mu|_{\mathrm{Dom}(f)}\big)$ is absolutely continuous with respect to $\mu|_{\mathrm{Im}(f)}$. When both f, f^{-1} are measurable and non-singular with respect to μ, we say that μ is *quasi-invariant* for f. A measurable set $S \subseteq V$ is called $\mathcal{F}{\downarrow_V}$*-invariant*, if $\langle \mathcal{F}{\downarrow_V} \rangle^+(S) \subseteq S$ up to a set of measure zero. Moreover, $\mathrm{IFS}(\mathcal{F}{\downarrow_V})$ is called *ergodic* with respect to μ, if μ is quasi-invariant for all the elements of $\mathcal{F}$, and there is no measurable $\mathcal{F}{\downarrow_V}$-invariant set S with $0 < \mu(S \cap V) < \mu(V)$.

Throughout the paper, the sets we are localizing the dynamics on are open subsets of smooth manifolds. We fix the Lebesgue measure on the manifolds and all the statements regarding ergodicity are with respect to the Lebesgue measure.

2.2. **Expanding maps.** For a linear map D, we denote its operator norm by $\|D\|$, and its co-norm by $m(D) := \inf\{|D(v)| : |v| = 1\}$. If D is invertible, then $m(D) = \|D^{-1}\|^{-1}$. A diffeomorphism $f \in \mathrm{Diff}^1_{\mathrm{loc}}(M)$ is called expanding, if there exists $\eta > 1$ such that $m(D_x f) > \eta$ for every $x \in \mathrm{Dom}(f)$. Clearly, the expanding property is $\mathcal{C}^1$-robust.

Definition 2.1. For $\eta > 1$ and $N \in \mathbb{N} \cup \{\infty\}$, we say a sequence $\{f_i\}_{i=1}^N$ in $\mathrm{Diff}^1_{\mathrm{loc}}(M)$ is *η-expanding at* $x_0 \in M$, if for any integer $i \in [1, N]$, $x_{i-1} \in \mathrm{Dom}(f_i)$ and $m(D_{x_{i-1}} f_i) > \eta$, where $x_i = f_i \circ \cdots \circ f_1(x_0)$. Furthermore, the sequence is *expanding at* x_0 if it is η-expanding for some $\eta > 1$.

2.3. **Quasi-conformality.** For a real number $\kappa \ge 1$, a matrix $D \in \mathrm{GL}(d, \mathbb{R})$ is *κ-conformal*, if $\|D\|/m(D) = \|D\| \cdot \|D^{-1}\| \le \kappa$ and a sequence $\{D_i\}_{i=1}^N$ in $\mathrm{GL}(d, \mathbb{R})$ is *κ-conformal*, if for any integer $n \in [1, N]$, $D_n D_{n-1} \cdots D_1$ is κ-conformal. Here, N can be finite or infinite. It follows immediately from definition that for $D_1, D_2 \in \mathrm{GL}(d, \mathbb{R})$, if D_i is κ_i-conformal ($i = 1, 2$), then $D_1 D_2$ is $\kappa_1\kappa_2$-conformal, and D_1^{-1} is κ_1-conformal. These in particular imply that for a κ-conformal sequence $\{D_i\}_{i=1}^N$, all the products of the form $D_j D_{j-1} \cdots D_i$ for $1 \le i < j \le N$ are κ^2-conformal. As derivatives of smooth maps are linear maps between tangent spaces, one can define similar notions for them.

Definition 2.2. For $\kappa \ge 1$ and $N \in \mathbb{N} \cup \{\infty\}$, we say a sequence $\{f_i\}_{i=1}^N$ in $\mathrm{Diff}^1_{\mathrm{loc}}(M)$ is *κ-conformal at* $x \in M$, if for any integer $n \in [1, N]$, $x \in \mathrm{Dom}(f^n)$ and the linear map $D_x f^n$ is κ-conformal, where $f^n = f_n \circ \cdots \circ f_1$. Furthermore, the sequence is *quasi-conformal at* x if it is κ-conformal at x for some $\kappa \geq 1$.

3. Expanding sequences

Here, we prove two technical results which will be used for showing ergodicity of quasi-conformal blenders. The first one provides a precise control of geometry under quasi-conformal expanding sequences. The second one is the standard bounded distortion lemma adapted to our setting of expanding sequences of local maps.

Throughout the section, M is a closed manifold of dimension d. For a sequence $\{f_i\}_{i=1}^{\infty}$ in $\mathrm{Diff}^{1+\alpha}_{\mathrm{loc}}(M)$, denote $f^i := f_i \circ \cdots \circ f_1$ and $f^0 := \mathrm{Id}$. We denote the open ball of radius of $r > 0$ around origin in $\mathbb{R}^d$ with $B_r(0)$.

3.1. **Control of geometry.** Our main goal in this subsection is to prove the following theorem.

Theorem 3.1. *Let $\{f_i\}_{i=1}^{\infty}$ be a sequence in $\mathrm{Diff}^{1+\alpha}_{\mathrm{loc}}(M)$ with bounded $\mathcal{C}^{1+\alpha}$ norm. Let also $x \in M$ and $\rho > 0$ be such that for any $n \in \mathbb{N}$, f_n is defined on $B(f^{n-1}(x), \rho)$. If the sequence is quasi-conformal expanding at x, then there exist $\xi_0 > 0$ and $\theta > 1$ such that for any $\xi \in (0, \xi_0]$ and $n \in \mathbb{N}$,*

$$f^n(B(x, r_n)) \subseteq B(f^n(x), \xi) \subseteq f^n(B(x, \theta r_n)),$$

for some $r_n > 0$.

In other words, one obtains a control of geometry of the iterations of a ball from certain assumptions on the derivatives at the center. The passage from linear to nonlinear follows from precise estimates on pseudo-orbits of corresponding product of matrices. The proof of this theorem occupies the entire subsection. As it is a local statement, we prove Theorem 3.1 by showing similar statements on Euclidean space and for uniformly contracting sequences of local diffeomorphisms.

Fix $R, C > 0, \kappa \geq 1 > \overline{\lambda} > \underline{\lambda} > 0$ and $\alpha \in (0, 1)$. For $N \in \mathbb{N} \cup \{\infty\}$, we consider the following hypotheses for the sequence $\{h_n\}_{n=1}^{N}$.

- (H0) $h_n : B_R(0) \to h_n(B_R(0))$ is a $\mathcal{C}^{1+\alpha}$ diffeomorphism fixing the origin,
- (H1) $\|h_n\|_{\mathcal{C}^{1+\alpha}} < C$,
- (H2) for any $y \in B_R(0)$, $\underline{\lambda} < m(D_y h_n) \leq \|D_y h_n\| < \overline{\lambda}$,
- (H3) h^n is κ-conformal at the origin.

Note that, it follows from (H2) that $h_n(\overline{B_R(0)}) \subseteq B_R(0)$.

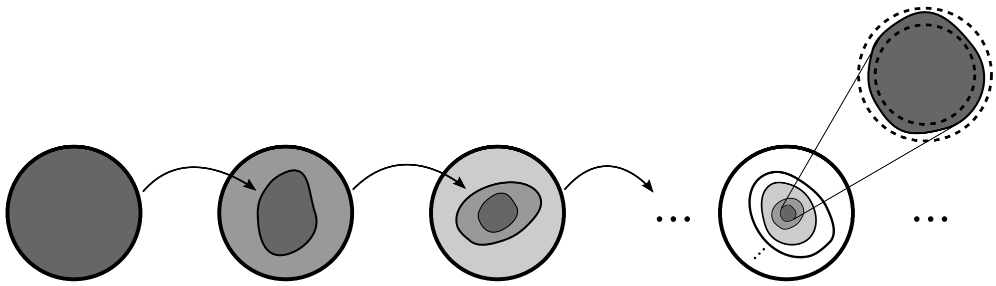


Figure 1. Sequence of maps satisfying (H0)-(H3) in Lemma 3.2.

Lemma 3.2. *Let $d \in \mathbb{N}$, $R, C > 0$, $\kappa \geq 1 > \overline{\lambda} > \underline{\lambda} > 0$ and $\alpha \in (0,1)$. Then, there exist $\xi_0 \in (0, R)$ and $\gamma, \theta > 1$ such that for any $\xi \in (0, \xi_0]$, any $n \in \mathbb{N}$ and any sequence $\{h_j\}_{j=1}^n$ of maps satisfying (H0)-(H3),*

$$B_{r/\theta}(0) \subseteq h^n(B_\xi(0)) \subseteq B_{\theta r}(0), \tag{2}$$

for $r = \xi |\det D_0 h^n|^{1/d}$. Moreover, for any $|x| \leq \xi$,

$$|h^n(x) - D_0 h^n(x)| < \gamma |\det D_0 h^n|^{1/d} |x|^{1+\alpha}. \tag{3}$$

This lemma follows from the next one on sequences of linear maps satisfying conditions (H2)-(H3). It establishes precise estimates on the difference between the orbits and certain pseudo-orbits.

Given a sequence $\{D_i\}_{i=1}^\infty$ in $\mathrm{GL}(d, \mathbb{R})$, for a pair $i < j$ of positive integers denote $D_{j,i} := D_j D_{j-1} \cdots D_i$.

Lemma 3.3 (Key Lemma)**.** *Let $C > 0$, $\kappa \geq 1 > \overline{\lambda} > \underline{\lambda} > 0$ and $\alpha \in (0,1)$. Then, there exist $\xi_1, \gamma > 0$ such that for any $n \in \mathbb{N}$, any sequence $\{D_i\}_{i=1}^n$ of matrices in $\mathrm{GL}(d, \mathbb{R})$ satisfying*

(C1) *Contraction: For any $1 \leq i \leq n$, $\underline{\lambda} \leq m(D_i) \leq \|D_i\| \leq \overline{\lambda} < 1$,*

(C2) *Quasi-conformality: For any $1 \leq i \leq n$, $\dfrac{\|D_{i,1}\|}{m(D_{i,1})} \leq \kappa$,*

and any sequence $\{y_i\}_{i=0}^n$ in $\mathbb{R}^d$ with $|y_0| \leq \xi_1$ and

$$|y_{i+1} - D_{i+1} y_i| < C |y_i|^{1+\alpha}, \tag{4}$$

the following holds

$$|y_n - D_{n,1} y_0| < \gamma |\det D_{n,1}|^{1/d} |y_0|^{1+\alpha}. \tag{5}$$

Proof. The proof of the lemma consists of several steps. First, a rough a priori upper bound for the norms of the terms in the sequence is given. Second, one observes that if the matrices in the sequence satisfy an extra assumption between the norm and the co-norm, a more accurate estimate holds which leads to the statement of the lemma. The extra assumption is not restrictive as it can be verified if one replaces the sequence by its large blocks of compositions.

For the proof, fix $C, \kappa, \overline{\lambda}, \underline{\lambda}, \alpha > 0$ and consider sequences $\{D_i\}_{i=1}^n$, $\{y_i\}_{i=0}^n$ as in the lemma. It follows from (C2) that for any $j > i$, $D_{j,i}$ is κ^2-conformal. Since $m(D_{j,i}) \leq |\det D_{j,i}|^{1/d} \leq \|D_{j,i}\|$, one gets

$$\frac{\|D_{j,i}\|}{|\det D_{j,i}|^{1/d}} \leq \frac{\|D_{j,i}\|}{m(D_{j,i})} \leq \kappa^2, \text{and } \frac{|\det D_{j,i}|^{1/d}}{m(D_{j,i})} \leq \frac{\|D_{j,i}\|}{m(D_{j,i})} \leq \kappa^2,$$

So the condition (C2) implies the following.

(C2)′ For any $j > i \geq 1$, $\kappa^{-2} |\det D_{j,i}|^{1/d} \leq m(D_{j,i}) \leq \|D_{j,i}\| \leq \kappa^2 |\det D_{j,i}|^{1/d}$.

Claim 1. *For any $\alpha' > 0$, there exist $K = K(\kappa, d, \overline{\lambda}, \underline{\lambda}, \alpha') \in \mathbb{N}$ and $\tau > 0$ such that for any $i \geq 0$,*

$$\left(\|D_{i+K,i+1}\| + \tau\right)^{1+\alpha'} \leq |\det D_{i+K,i+1}|^{1/d}. \tag{6}$$

Proof. For $t \in \mathbb{R}^+$, denote $\varphi(t) := t^{\frac{1}{1+\alpha'}} - \kappa^2 t$. Since $1+\alpha' > 1$, there exists $T = T(\alpha', \kappa) > 0$, such that $\varphi(t)$ is positive and increasing on $(0, T)$. Let $K \in \mathbb{N}$ be large enough such that $\overline{\lambda}^K < T$ and $\tau := \varphi(\kappa^{-2}\underline{\lambda}^K)$. Now, the conclusion easily follows from (C2)$'$ and (C1). □

Note that from (4), $|y_{i+1}| < C|y_i|^{1+\alpha} + \|D_{i+1}\||y_i|$. Therefore, if $|y_0| \le \xi' := C^{-\frac{1}{\alpha}}(1-\overline{\lambda})^{\frac{1}{\alpha}}$, then for any $i \ge 0$, $C|y_i|^\alpha + \|D_{i+1}\| \le 1$ and so

$$|y_{i+1}| \le |y_i|. \tag{7}$$

Define $\epsilon_i := y_{i+1} - D_{i+1}y_i \in \mathbb{R}^d$. For any pair $i, k \ge 0$, one obtains an explicit formula for y_{i+k} in terms of y_i and $\{\epsilon_j : i \le j < i+k\}$,

$$\begin{aligned} y_{i+k} &= D_{i+k}y_{i+k-1} + \epsilon_{i+k-1} = D_{i+k}(D_{i+k-1}y_{i+k-2} + \epsilon_{i+k-2}) + \epsilon_{i+k-1} = \cdots \\ &= D_{i+k,i+1}y_i + \sum_{j=i+2}^{i+k} D_{i+k,j}\epsilon_{j-2} + \epsilon_{i+k-1}. \end{aligned} \tag{8}$$

Claim 2. *If $|y_0| \le \xi'$, then for any $i \ge 0$ and $k \ge 1$,*

$$|D_{i+k,i+1}y_i - y_{i+k}| < Ck|y_i|^{1+\alpha}.$$

Proof. By (8) and since $\|D_j\| < 1$,

$$|y_{i+k} - D_{i+k,i+1}y_i| \le \sum_{j=i}^{i+k-1} |\epsilon_j|.$$

On the other hand, (4) and (7) imply that for any $j = i, \ldots, i+k-1$, $|\epsilon_j| < C|y_j|^{1+\alpha} \le C|y_i|^{1+\alpha}$. This finishes the proof of Claim 2. □

Now, let $\alpha' \in (0, \alpha)$. Suppose that $K \in \mathbb{N}$, $\tau > 0$ are the numbers provided by Claim 1. The aim of Claims 3 and 4 is to prove the conclusion of Lemma 3.3 when n is divisible by K, and then Claim 5 completes the proof for arbitrary n.

Claim 3. *There exist $\xi_1, \tau' > 0$ with $\tau' < \min\{1 - \overline{\lambda}^K, \tau\}$ and $\xi_1 \le \xi'$ such that if $|y_i| \le \xi_1$, then for any $p \ge 1$,*

$$|y_{pK+i}| \le \prod_{j=0}^{p-1} \left(\|D_{(j+1)K+i,jK+i+1}\| + \tau'\right)|y_i| \le |y_i|. \tag{9}$$

Proof. It suffices to prove (9) for $p = 1$. The general case follows immediately from induction on p. Let $\tau', \xi_1 > 0$ be such that $\tau' < \min\{1 - \overline{\lambda}^K, \tau\}$ and $\xi_1 \le \min\{(C^{-1}K^{-1}\tau')^{\frac{1}{\alpha}}, \xi'\}$. Then, by Claim 2,

$$\frac{|y_{i+K}|}{|y_i|} < \|D_{i+K,i+1}\| + CK|y_i|^\alpha \le \|D_{i+K,i+1}\| + \tau' \le \overline{\lambda}^K + \tau' < 1.$$

This finishes the proof of Claim 3. □

Claim 4. *There exists $\gamma' > 0$ such that for any $p \in \mathbb{N}$ and $q \ge 0$,*

$$|y_{pK+q} - D_{pK+q,q+1}y_q| < \gamma' |\det D_{pK+q,q+1}|^{1/d} |y_q|^{1+\alpha},$$

provided that $|y_q| \le \xi$.

Proof. Define $\alpha'' := \alpha - \alpha' > 0$. For simplicity, for a fixed q and $1 \leq i \leq p$, we write $D'_i := D_{iK+q,(i-1)K+q+1}$, $\overline{\lambda}'_i := \|D'_i\|$, $\underline{\lambda}'_i := m(D'_i)$. Moreover, for $0 \leq i \leq p$, let $y'_i := y_{iK+q}$. Also, denote $\epsilon'_i := y'_{i+1} - D'_{i+1}y'_i$. Similar to (8), one gets,

$$y'_p = D'_{p,1}y'_0 + D_{p,2}\epsilon'_0 + \cdots + D'_{p,p-1}\epsilon'_{p-3} + D'_p\epsilon'_{p-2} + \epsilon'_{p-1},$$

and so,

$$|y'_p - D'_{p,1}y'_0| \leq |D'_{p,2}\epsilon'_0| + \cdots + |D'_p\epsilon'_{p-2}| + |\epsilon'_{p-1}|. \tag{10}$$

Let $\overline{\lambda}' := \overline{\lambda}^K$ and $\underline{\lambda}' := \underline{\lambda}^K$. Claim 1 and $\tau' \leq \tau$ imply

$$(\overline{\lambda}'_j + \tau')^{1+\alpha} = (\overline{\lambda}'_j + \tau')^{1+\alpha'}(\overline{\lambda}'_j + \tau')^{\alpha''} \leq \delta'_j(\overline{\lambda}' + \tau')^{\alpha''}. \tag{11}$$

Hence, by Claims 2 and 3,

$$\begin{aligned} |\epsilon'_k| \leq CK|y'_k|^{1+\alpha} &\leq CK|y'_0|^{1+\alpha}\prod_{j=1}^{k}(\overline{\lambda}'_j + \tau')^{1+\alpha} \\ &\leq CK(\overline{\lambda}' + \tau')^{k\alpha''}\delta'_k\cdots\delta'_1|y'_0|^{1+\alpha} \end{aligned}$$

By (C2)$'$, for $p \geq k+2$,

$$\begin{aligned} |D'_{p,k+2}\epsilon'_k| &\leq \kappa^2\delta'_p\cdots\delta'_{k+2}|\epsilon'_k| \\ &\leq CK\kappa^2\delta'_p\cdots\delta'_{k+2}(\overline{\lambda}' + \tau')^{k\alpha''}\delta'_k\cdots\delta'_1|y'_0|^{1+\alpha} \\ &\leq \frac{CK\kappa^2}{\underline{\lambda}'}(\overline{\lambda}' + \tau')^{k\alpha''}\delta'_p\cdots\delta'_1|y'_0|^{1+\alpha}, \end{aligned}$$

where the last inequality follows from $\delta'_{k+1} \geq \underline{\lambda}'_{k+1}$. Now, from (10),

$$|y'_p - D'_{p,1}y'_0| \leq \frac{CK\kappa^2}{\underline{\lambda}'}\Big(\sum_{k=0}^{p-1}(\overline{\lambda}' + \tau')^{k\alpha''}\Big)\delta'_p\cdots\delta'_1|y'_0|^{1+\alpha}.$$

On the other hand, $\overline{\lambda}' + \tau' < 1$. Consequently,

$$\sum_{k=0}^{p-1}(\overline{\lambda}' + \tau')^{k\alpha''} \leq C' := \sum_{k=0}^{\infty}(\overline{\lambda}' + \tau')^{k\alpha''} < \infty.$$

Therefore, taking $\gamma' := \kappa^2 CKC'\underline{\lambda}^{-K}$ finishes the proof. □

Claim 5. *There exists $\gamma > 0$ such that if $|y_0| \leq \xi_1$,*

$$|y_n - D_{n,1}y_0| < \gamma|\det D_{n,1}|^{1/d}|y_0|^{1+\alpha}.$$

Proof. Let $\delta_i := |\det D_i|^{1/d}$ and write $n = pK + q$ for $0 \leq q < K$. Since $|y_q| \leq \xi_1$ from Claim 4,

$$|y_{pK+q} - D_{pK+q,q+1}y_q| \leq \gamma'\delta_{pK+q}\cdots\delta_{q+1}|y_q|^{1+\alpha}.$$

Meanwhile, from (C2)$'$ and Claim 2, it follows that

$$\begin{aligned} |D_{pK+q,q+1}y_q - D_{pK+q,1}y_0| &\leq \|D_{pK+q,q+1}\|.|y_q - D_{q,1}y_0| \\ &\leq \kappa^2\delta_{pK+q}\cdots\delta_{q+1}Cq|y_0|^{1+\alpha}. \end{aligned}$$

Finally, by Claim 4 and since $\delta_i \geq \underline{\lambda}$, one has

$$\begin{aligned}|y_{pK+q} - D_{pK+q,1}y_0| &\leq \delta_{pK+q}\cdots\delta_{q+1}|y_0|^{1+\alpha}(\gamma' + \kappa^2 Cq)\\ &\leq \delta_{pK+q}\cdots\delta_{q+1}\delta_q\cdots\delta_1|y_0|^{1+\alpha}\underline{\lambda}^{-q}(\gamma' + \kappa^2 Cq).\end{aligned}$$

So, the conclusion holds for $\gamma := \underline{\lambda}^{-K}(\gamma' + \kappa^2 CK)$. □

Claim 5 completes the proof of Lemma 3.3. □

Now, we are ready to complete the proof of Lemma 3.2 and Theorem 3.1.

Proof of Lemma 3.2. Take $x \in B_R(0)$. Clearly, if the sequence $\{h_j\}_{j=1}^n$ satisfies (H0)-(H3), then the conditions of Lemma 3.3 are satisfied for $D_j := D_0 h_j$, $y_j := h^j(x) = h_i(x_{i-1})$. So, (3) holds for any $|x| \leq \xi_1$. In order to prove (2), note that (3) in particular implies that

$$|D_0h^n(x)| - \gamma|\det D_0h^n(x)|^{1/d}|x|^{1+\alpha} < |h^n(x)|, \tag{12}$$

and

$$|h^n(x)| < |D_0h^n(x)| + \gamma|\det D_0h^n(x)|^{1/d}|x|^{1+\alpha}. \tag{13}$$

On the other hand, by (C2)$'$,

$$\kappa^{-2}|\det D_0h^n(x)|^{1/d}|x| \leq |D_0h^n(x)| \leq \kappa^2|\det D_0h^n(x)|^{1/d}|x|.$$

This combined with (12) and (13), implies that

$$\kappa^{-2}|x| - \gamma|x|^{1+\alpha} < \frac{|h^n(x)|}{|\det D_0h^n(x)|^{1/d}} < \kappa^2|x| + \gamma|x|^{1+\alpha}.$$

Therefore, for any ξ with $\kappa^{-2} - \gamma\xi^\alpha > 0$, $B_{r_1(\xi)}(0) \subseteq h^n(B_\xi(0)) \subseteq B_{r_2(\xi)}(0)$, provided that

$$r_1(\xi) := (\kappa^{-2} - \gamma\xi^\alpha)\xi|\det D_0h^n|^{1/d}, \quad r_2(\xi) := (\kappa^2 + \gamma\xi^\alpha)\xi|\det D_0h^n|^{1/d}.$$

To prove (2), take $\theta > \kappa^2$. Then, for sufficiently small $\xi > 0$, $\kappa^2 + \gamma\xi^\alpha < \theta$ and $\kappa^{-2} - \gamma\xi^\alpha > \theta^{-1}$. □

Proof of Theorem 3.1. Let $R_1 > 0$ be smaller than the radius of injectivity of the exp function on M and $x_j := f^j(x)$. Suppose that the sequence is κ-conformal η-expanding at x_0. Since the sequence has bounded $\mathcal{C}^{1+\alpha}$ norm, there is $\rho' \in (0, \rho)$ and $\overline{\eta}, \underline{\eta} > 1$ such that for any $i \in \mathbb{N}$ and $y \in B(x_{i-1}, \rho')$,

$$\underline{\eta} < m(D_y f_i) \leq \|D_y f_i\| < \overline{\eta}.$$

In fact, one has $\sup_{i\in\mathbb{N}} \|Df_i|_{B(x_{i-1},\rho)}\| < \infty$, and if $C > \sup_{i\in\mathbb{N}} \|f_i\|_{\mathcal{C}^{1+\alpha}}$,

$$|m(D_{x_{i-1}}f_i) - m(D_y f_i)| < C|x_{i-1} - y|^\alpha.$$

So, $m(D_y f_i) > \underline{\eta}$ for any $y \in B(x_{i-1}, \rho')$, provided that $\rho' < (C^{-1}(\eta - \underline{\eta}))^{\frac{1}{\alpha}}$, .

For $R < R_0 := (\overline{\eta})^{-1}\min\{R_1, \rho'\}$ and $i \in \mathbb{N}$, the map $\tilde{f}_i := \exp_{x_i}^{-1} \circ f_i \circ \exp_{x_{i-1}}$ is defined on $B_R(0) \subseteq T_{x_i}(M)$ and is a diffeomorphism onto its image. After an isometric identification of the tangent spaces with $\mathbb{R}^d$, one can consider the sequence $\{\tilde{f}_i\}_{i=1}^\infty$ as a sequence of expanding maps defined on $B_R(0) \subseteq \mathbb{R}^d$. By uniform expansion of the maps, $B_R(0) \subseteq \tilde{f}_i(B_R(0))$.

Next, fix $n \in \mathbb{N}$. The sequence $\{h_j\}_{j=1}^n$ defined by $h_j := \tilde{f}_{n+1-j}^{-1}|_{B_R(0)}$ satisfies hypotheses (H0)-(H3) with constants independent of the choice of n. So, by Lemma 3.2, there are $\theta > 1$ and $\xi_0 > 0$ such that for any $\xi \leq \xi_0$ and for some $r_n > 0$,

$$(h^n)^{-1}(B_{r_n}(0)) \subseteq B_\xi(0) \subseteq (h^n)^{-1}(B_{\theta r_n}(0)) \subseteq B_R(0).$$

This finishes the proof, since $(h^n)^{-1} = \exp_{x_n}^{-1} \circ f^n \circ \exp_{x_0}$ and for small $r > 0$, the function $\exp_x : T_xM \to M$, maps $B_r(0) \subseteq T_xM$ to $B(x,r) \subseteq M$. □

3.2. **Bounded distortion.** In this subsection, we present a bounded distortion lemma for a sequence of contracting maps which permits us to control the growth of measure of iterations of measurable sets. The proof here is an adaptation of the classical argument to our setting. Let $R_1 > 0$ be smaller than the radius of injectivity of the exp function on M.

Lemma 3.4. *Let $\alpha, \lambda \in (0,1)$ and $C > 0$. Then, there exists $L > 1$ such that for any $R < R_1$, any $n \in \mathbb{N}$, any sequence $\{x_j\}_{j=0}^n$ in M, and any sequence $\{h_j\}_{j=1}^n$ in $\mathrm{Diff}^{1+\alpha}_{\mathrm{loc}}(M)$ with $h_j : B(x_{j-1}, R) \to h_j(B(x_{j-1}, R))$ satisfying $h_j(x_{j-1}) = x_j$, $\|Dh_j\| < \lambda$, $\|h_j\|_{\mathcal{C}^{1+\alpha}} < C$, and every pair of measurable sets $S_1, S_2 \subseteq B(x_0, R)$ of positive Lebesgue measure,*

$$L^{-1}\frac{\mathrm{Leb}(S_1)}{\mathrm{Leb}(S_2)} < \frac{\mathrm{Leb}(h^n(S_1))}{\mathrm{Leb}(h^n(S_2))} < L\frac{\mathrm{Leb}(S_1)}{\mathrm{Leb}(S_2)}.$$

Recall that $h^n := h_n \circ \cdots \circ h_1$.

Proof. Since there is an upper bound for the $\mathcal{C}^{1+\alpha}$ norm of the derivative of the exp function on the balls of radius R_1 on the whole manifold M, by replacing h_j with $\exp_{x_j} \circ h_j \circ \exp_{x_{j-1}}^{-1} : B_R(0) \to B_R(0)$, one can assume that the maps are defined between open sets of $\mathbb{R}^d$. Now, it is enough to show that there exists $L_1 > 1$ such that for any measurable set $S \subseteq B_R(0)$,

$$L_1^{-1}|\det D_0h^n|\mathrm{Leb}(S) \leq \mathrm{Leb}(h^n(S)) \leq L_1|\det D_0h^n|\mathrm{Leb}(S). \tag{14}$$

Since the sequence has bounded $\mathcal{C}^{1+\alpha}$ norm, the maps $z \mapsto \log|\det D_zh_j|$ are α-Hölder on $B_R(0)$ with some uniform constant, that is, there exists $L' > 0$ (independent of j) such that for any $j \geq 1$ and any pair $z, z' \in B_R(0)$,

$$\Big|\log\big|\det D_zh_j\big| - \log\big|\det D_{z'}h_j\big|\Big| < L'|z - z'|^\alpha.$$

For $x \in B_R(0)$ and $j \leq n$, denote $x_j := h^j(x)$. From the contraction property,

$$|x_j| \leq \Big(\sup_{z\in B_R(0)} \|D_zh^j\|\Big)|x| \leq \lambda^j|x| \leq \lambda^j R.$$

Therefore,

$$\begin{aligned}\Big|\log\frac{|\det D_xh^n|}{|\det D_0h^n|}\Big| &= \sum_{j=0}^{n-1}\Big|\log|\det D_{x_j}h_{j+1}| - |\det D_0h_{j+1}|\Big| \\ &< L'\sum_{j=0}^{n-1}|x_j|^\alpha \leq L'R^\alpha\sum_{j=0}^{n-1}\lambda^{j\alpha}.\end{aligned}$$

Now, since $\overline{\lambda} < 1$, $L_1 := \exp\left(L' R^\alpha \sum_{j=0}^{\infty} \lambda^{j\alpha}\right) < \infty$ and so

$$L_1^{-1} |\det D_0 h^n| \le |\det D_x h^n| \le L_1 |\det D_0 h^n|. \tag{15}$$

By the change of variable formula,

$$\int_S |\det D_x h^n| \mathrm{dLeb}(x) \le L_1 |\det D_0 h^n| \mathrm{Leb}(S).$$

The proof of the other inequality in (14) is similar. □

3.3. **Infiltrated quasi-conformality.** In this subsection, we show that under hypotheses (H0)-(H3), quasi-conformality of the derivatives at the origin leads to the quasi-conformality in a neighbourhood. Informally, the idea is that the contracting assumption forces the derivatives of long blocks in the nearby points to imitate the behaviour of derivatives at the origin. The results of this subsection will not be used in other parts of the paper and is included here for its own interest.

Theorem 3.5. *Let $x \in M$, $\{f_i\}_{i=1}^\infty$ be a sequence in $\mathrm{Diff}^{1+\alpha}(M)$ with bounded $\mathcal{C}^{1+\alpha}$ norm and quasi-conformal expanding at x. Then, there exist $R > 0$ and $\theta > 1$ such that for any $n \in \mathbb{N}$ and any ball $B(y, r) \subseteq B(f^n(x), R)$,*

$$B(f^{-n}y, r_n) \subseteq f^{-n}(B(y, r)) \subseteq B(f^{-n}(y), \theta r_n),$$

where $f^{-n} := (f_n \circ \cdots \circ f_1)^{-1}$ and $r_n = r\theta^{-\frac{1}{2}} |\det D_x f^{-n}|^{1/d}$.

To prove Theorem 3.5, we first prove the following proposition.

Proposition 3.6. *Let $R, C > 0$, $\kappa \ge 1 > \overline{\lambda} > \underline{\lambda} > 0$ and $\alpha \in (0, 1)$. Then, there exist $\xi_0 > 0$, $\overline{\kappa} > 1$ such that any sequence $\{h_j\}_{j=1}^\infty$ of maps satisfying hypotheses (H0)-(H3) is $\overline{\kappa}$-conformal at every point of $B_{\xi_0}(0)$.*

Proof. The proposition follows from a refinement of the proof of Lemma 3.3. For $\alpha < 1$, one should take blocks of compositions and repeat the claims of the proof of Lemma 3.3. To avoid repeating the arguments, here we give a proof for $\mathcal{C}^{1+\mathrm{Lip}}$ regularity, that is, for $\alpha = 1$. For $x \in B_R(0)$, denote $x_i := h^i(x)$, $D_i := D_0 h_i$, $\delta_i := |\det D_i|^{1/d}$, $\tilde{D}_i := D_{x_{i-1}} h_i$ and $\mathcal{E}_i := D_i - \tilde{D}_i$. Observe that for $n \in \mathbb{N}$,

$$D_0 h^n - D_x h^n = D_n \cdots D_1 - \tilde{D}_n \cdots \tilde{D}_1 = \sum_{i=1}^{n} D_n \cdots D_{i+1} \mathcal{E}_i \tilde{D}_{i-1} \cdots \tilde{D}_1.$$

Hence,

$$\|D_x h^n - D_0 h^n\| \le \sum_{i=1}^{n} \|D_n \cdots D_{i+1}\| \cdot \|\mathcal{E}_i\| \cdot \|\tilde{D}_{i-1} \cdots \tilde{D}_1\|. \tag{16}$$

Since $D_n \cdots D_{i+1}$ is κ^2-conformal, in view of (C2)$'$, one gets that $\|D_n \cdots D_{i+1}\| \le \kappa^2 \delta_n \cdots \delta_{i+1}$. On the other hand, (H1) implies $\|\tilde{D}_{i-1} \cdots \tilde{D}_1\| \le \overline{\lambda}^{i-1}$. Now, for $\theta > 1$ given by Lemma 3.2, $|x_{i-1}| \le \theta \delta_{i-1} \cdots \delta_1 |x|$ and so, by (H3),

$$\|\mathcal{E}_i\| = |D_{x_{i-1}} h_i - D_0 h_i| \le C\theta |x_{i-1}| \le C\theta R \delta_{i-1} \cdots \delta_1.$$

Using (16),

$$\|D_xh^n - D_0h^n\| \le \sum_{i=1}^{n} CR\kappa^2\delta_n\cdots\delta_{i+1}\delta_{i-1}\cdots\delta_1\overline{\lambda}^{i-1}$$
$$\le CR\kappa^2\underline{\lambda}^{-1}\Big(\sum_{i=1}^{n}\overline{\lambda}^{i-1}\Big)\delta_n\cdots\delta_1.$$

Now, by the convergence of the series $\sum_{i=0}^{\infty}\overline{\lambda}^{i-1}$, there exists $C_1 > 0$ such that for any $n \ge 1$ and $x \in B_R(0)$,

$$\|D_xh^n\| \le \|D_0h^n\| + \|D_xh^n - D_0h^n\| < C_1|\det D_0h^n|^{1/d}. \tag{17}$$

By (15), one obtains that $\|D_xh^n\| \le C_1L_1^{1/d}|\det D_xh^n|^{1/d}$. Thus, there exists $\overline{\kappa} = \overline{\kappa}(d, C_1L_1^{1/d}) > 0$ such that D_xh^n is $\overline{\kappa}$-conformal, as claimed. □

Proof of Theorem 3.5. The proof is similar to the one of Theorem 3.1. Take $x_i, \overline{\eta}, \underline{\eta}, R$ as in the proof of that theorem. Define $\{y_i\}_{i=0}^n$ by $y_n := y$ and $y_{i-1} := f_i^{-1}(y_i)$. By uniform expansion, one obtains that for any $j < n$,

$$f_{j+1}^{-1}\circ\cdots\circ f_n^{-1}(B(y_n, r)) \subseteq B(y_j, r)\cap B(x_j, R).$$

Now, define $\hat{f}_i := (\exp_{y_i}^{-1}\circ f_i\circ \exp_{y_{i-1}})|_{B_r(0)}$ and $\hat{h}_i := \hat{f}_{n+1-i}^{-1}|_{B_r(0)}$. So, the conclusion follows from Lemma 3.2 for this sequence. Indeed, condition (H3) is guaranteed by Proposition 3.6. □

Remark 3.7. By Proposition 3.6, Lemma 3.4 and the results in the theory of quasi-conformal and quasi-symmetric maps, one can give another proof for Theorem 3.5. In fact, if a map is κ-conformal on its domain, then there exists a bound for the ratio between outer and inner radii of the image of a ball (see [HK98, Section 4] and [Väi89]). Then, the inner and outer radii can be estimated by means of estimating the volume and the bounded distortion lemma (Lemma 3.4).

4. Quasi-conformal dynamics

This section is devoted to the notion of covering property for derivatives. We discuss its consequences in providing quasi-conformal orbit-branches and also sufficient conditions to ensure it.

Throughout the section, M is a boundaryless Riemannian manifold of dimension d. We consider the fiber bundle $\pi : \mathcal{E}(M) \to M$. Recall that for $\mathbf{w} = (x, (v_1, \dots, v_d)) \in \mathcal{E}(M)$, $\|\mathbf{w}\|_2$ is defined by $\|\mathbf{w}\|_2 = (\sum_i |v_i|^2)^{\frac{1}{2}}$, where $|.|$ is the norm induced by the Riemannian metric on TM.

Furthermore, for a linear map $T : W_0 \to W_1$ between finite dimensional vector spaces endowed with inner products, we denote the Hilbert-Schmidt norm of T by $\|T\|_2$, defined by

$$\|T\|_2 := \Big(\sum_i |Te_i|^2\Big)^{\frac{1}{2}},$$

where $\{e_i\}_i$ is an orthonormal basis for W_0. In particular, for $f \in \mathrm{Diff}^1_{\mathrm{loc}}(M)$, and $\mathbf{w} = (x, (e_1, \dots, e_d)) \in \mathcal{E}(M)$, where $x \in \mathrm{Dom}(f)$ and $\{e_1, \dots, e_d\}$ form an orthonormal basis for T_xM, it follows that $\|\hat{D}_xf\|_2 = \|\hat{\mathcal{D}}f(\mathbf{w})\|_2$.

For linear isomorphisms $T : W_0 \to W_1$ and $S : W_1 \to W_2$ between d-dimensional vector spaces, we will use the following properties.

$$\|T\| \le \|T\|_2 \le \sqrt{d}\|T\|. \tag{18}$$

$$\|S \circ T\|_2 \le \|S\|.\|T\|_2 \text{ and } \|S \circ T\|_2 \le \|S\|_2.\|T\|. \tag{19}$$

$$\text{If } |\det T| = 1, \text{ then } \|T^{-1}\| \le \|T\|^{d-1}. \text{ Therefore, } T \text{ is } \|T\|^d\text{-conformal.} \tag{20}$$

4.1. **Stable quasi-conformality.** In this subsection, we present a criterion for the existence of quasi-conformal orbit-branches in pseudo-semigroup actions.

A subset $\mathcal{W} \subseteq \mathcal{E}(M)$ is called *bounded*, if $\sup_{\mathbf{w}\in\mathcal{W}} \|\mathbf{w}\|_2 < \infty$.

Theorem 4.1 (Quasi-conformality criterion)**.** *Let $V \subseteq M$ be an open set and $\mathcal{F} \subseteq \mathrm{Diff}^1_{\mathrm{loc}}(M)$. Then, the following are equivalent*

(a) *There exists $\kappa > 1$ such that for every $x \in V$, the pseudo-semigroup generated by $\mathcal{F}\!\downarrow_V$ has a κ-conformal orbit-branch at x.*

(b) *There exists a bounded subset $\mathcal{W} \subseteq \mathcal{E}(M)$ such that $\pi(\mathcal{W}) = V$, and*

$$\mathcal{W} \subseteq \bigcup_{f\in\mathcal{F}} (\hat{\mathcal{D}}f)^{-1}(\mathcal{W}). \tag{21}$$

Proof. For the implication (a)⇒(b). For every $x \in V$, let $\{f_{i,x}\}_{i=1}^{\infty}$ be a sequence defining a κ-conformal orbit-branch of $\mathcal{F}\!\downarrow_V$ at x. Consider an arbitrary orthonormal basis $\{e_i\}_i$ for T_xM and let $\mathbf{w}_x = (x, \mathbf{e}) \in \pi^{-1}(x)$ be such that $\mathbf{e} = (e_1, \dots, e_d)$. Note that for any $x \in V$, $\|\mathbf{w}_x\|_2 = \sqrt{d}$. Then, the set $\mathcal{W}$ defined by

$$\mathcal{W} := \bigcup_{x\in V} \bigcup_{i\ge 0} \hat{\mathcal{D}}f^i_x(\mathbf{w}_x),$$

satisfies the desired properties, where $f^i_x = f_{i,x} \circ f_{i-1,x} \circ \cdots \circ f_{1,x}$. Indeed, $\pi(\mathcal{W}) = V$ and for any $x \in V$ and $i \in \mathbb{N}$, $\hat{D}_x f^i_x$ is κ-conformal and so by (19),

$$\|\hat{\mathcal{D}}f^i_x(\mathbf{w}_x)\|_2 \le \|\hat{D}_x f^i_x\|.\|\mathbf{w}_x\|_2 \le \kappa\sqrt{d}.$$

Therefore, $\sup_{\mathbf{w}\in\mathcal{W}} \|\mathbf{w}\|_2 \le \kappa\sqrt{d}$ and so $\mathcal{W}$ is bounded. On the other hand, covering property (21) follows immediately from the definition of $\mathcal{W}$, since every element of $\mathcal{W}$ is of the following form

$$\hat{\mathcal{D}}f^i_x(\mathbf{w}_x) = (\hat{\mathcal{D}}f_{i+1,x})^{-1}(\hat{\mathcal{D}}f^{i+1}_x(\mathbf{w}_x)) \in (\hat{\mathcal{D}}f_{i+1,x})^{-1}(\mathcal{W}).$$

Next, in order to show (b)⇒(a), first denote $H := \sup_{\mathbf{w}\in\mathcal{W}} \|\mathbf{w}\|_2$. Fix $\mathbf{w} = (x, (v_1, \dots, v_d)) \in \mathcal{W}$. We first claim that there exists a sequence $\{f_i\}_{i=1}^{\infty}$ in $\mathcal{F}$ such that for any $n \ge 1$, $\hat{\mathcal{D}}f^n(\mathbf{w}) \in \mathcal{W}$. The proof is by induction, assume that $f_1, \dots, f_n$ are defined satisfying the properties. Then,

$$\hat{\mathcal{D}}f^n(\mathbf{w}) \in \mathcal{W} \subseteq \bigcup_{f\in\mathcal{F}} (\hat{\mathcal{D}}f)^{-1}(\mathcal{W}).$$

So, there is $f_{n+1} \in \mathcal{F}$ with $\hat{\mathcal{D}}f^n(\mathbf{w}) \in (\hat{\mathcal{D}}f_{n+1})^{-1}(\mathcal{W})$. Consequently, by the chain rule, $\hat{\mathcal{D}}f^{n+1}(\mathbf{w}) \in \mathcal{W}$. This finishes the proof of the claim.

Consider an orthonormal basis $\{e_1, \dots, e_d\}$ for T_xM and let $T : T_xM \to T_xM$ be the linear map with $T(e_i) = v_i$. Clearly, $|\det T| = 1$, $\|T\|_2 = \|\mathbf{w}\|_2 \le H$, and for any $n \ge 0$, $\|\hat{D}_x f^n \circ T\|_2 = \|\hat{\mathcal{D}}f^n(\mathbf{w})\|_2 \le H$. The last inequality holds since $\hat{\mathcal{D}}f^n(\mathbf{w}) \in \mathcal{W}$. Using (18) and (20), one obtains

$$\|\hat{D}_x f^n\| \le \|\hat{D}_x f^n \circ T\|.\|T^{-1}\| \le \|\hat{D}_x f^n \circ T\|.\|T\|^{d-1} \le H^d.$$

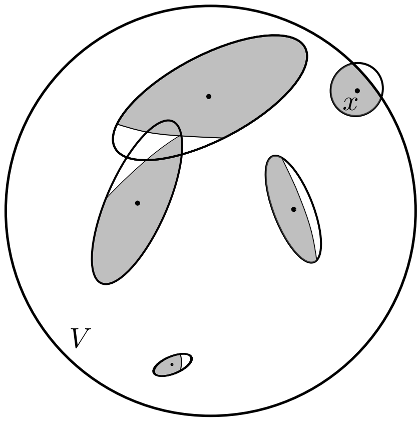


FIGURE 2. Covering condition (22): diversity of maps corresponding to $x \in V$, i.e. maps in the subfamily $\mathcal{F}_x$.

Again, by (20), this implies that the sequence $\{f_i\}_{i=1}^{\infty}$ is κ-conformal at x for $\kappa = H^{d^2}$. ☐

Corollary 4.2 (Stable quasi-conformality)**.** *Let $V \subseteq M$ be an open set and $\mathcal{F} \subseteq \mathrm{Diff}^1_{\mathrm{loc}}(M)$. Assume that $\mathcal{W}$ is an open subset of $\mathcal{E}(M)$ with compact closure satisfying $\pi(\mathcal{W}) = V$ and*

$$\overline{\mathcal{W}} \subseteq \bigcup_{f \in \mathcal{F}} (\hat{\mathcal{D}} f)^{-1}(\mathcal{W}). \tag{22}$$

Then, there exists $\kappa > 1$ such that for any family $\tilde{\mathcal{F}}$ in a $\mathcal{C}^1$ neighbourhood of $\mathcal{F}$, $\mathrm{IFS}(\tilde{\mathcal{F}}{\downarrow_V})$ has a κ-conformal orbit-branch at every point of V.

Proof. By the compactness of $\overline{\mathcal{W}}$ and the openness of $\mathcal{W}$, the same covering property (22) holds for any family $\tilde{\mathcal{F}}$ in a small $\mathcal{C}^1$ neighbourhood of $\mathcal{F}$. Hence, the conclusion follows from Theorem 4.1. Moreover, the proof of Theorem 4.1 shows that $\kappa = (\sup_{\mathbf{w} \in \mathcal{W}} \|\mathbf{w}\|_2)^{d^2}$ only depends on $\mathcal{W}$. ☐

Remark 4.3. For oriented manifolds and orientation-preserving maps, one can work with the following alternative fiber bundle instead of $\mathcal{E}(M)$,

$$\{(x, \mathbf{v}) : x \in M,\ \mathbf{v} \in (T_x M)^d,\ \text{and } \omega|_x(\mathbf{v}) = 1\},$$

where ω is the volume form on M induced from the Riemannian metric compatible with the orientation, and $\omega|_x$ is its restriction to $T_x M$. This defines a $\mathrm{SL}(d, \mathbb{R})$ fiber bundle over M which is a quotient of $\mathcal{E}(M)$ by an involution.

Reinterpretation of the covering property. Let us discuss the meaning of covering condition (22). Indeed, it is equivalent to the following (see Figure 2):

- *For any $x \in \overline{V}$, there exists $\mathcal{F}_x \subseteq \mathcal{F}$ such that*
  (i) *$f(x) \in V$, for $f \in \mathcal{F}_x$,*
  (ii) $\overline{\mathcal{W}_x} \subseteq \bigcup_{f \in \mathcal{F}_x} (\hat{\mathcal{D}} f)^{-1}(\mathcal{W}_{f(x)})$, where $\mathcal{W}_x := \pi^{-1}(x) \cap \mathcal{W}$.

Roughly speaking, this means that over every point x there are several maps in the family $\mathcal{F}$ with diverse directions of contraction and expansion for the normalized derivative that allows to obtain covering (ii) which yields the quasi-conformality along an orbit-branch of x.

In the case of $M = \mathbb{R}^d$, $\mathcal{E}(\mathbb{R}^d)$ is isomorphic to the trivial fiber bundle $\mathbb{R}^d \times \mathrm{SL}^{\pm}(d,\mathbb{R})$ and the action of $\hat{\mathcal{D}}f$ on fibers in nothing but the product of matrices in $\mathrm{SL}^{\pm}(d,\mathbb{R})$. More precisely, for $f \in \mathrm{Diff}^1_{\mathrm{loc}}(\mathbb{R}^d)$, $\hat{\mathcal{D}}f$ maps $(x, A) \in \mathbb{R}^d \times \mathrm{SL}^{\pm}(d,\mathbb{R})$ to $(f(x), A_x A)$ where $A_x := \hat{D}_x f \in \mathrm{SL}^{\pm}(d,\mathbb{R})$. In other words, the covering condition (22) will be reduced to finding a map $f \in \mathcal{F}$ such that $A_x A$ is in the bounded set $\mathcal{W}_{f(x)}$. Observe that $\mathbb{R}^d \times \mathrm{SL}(d,\mathbb{R})$ is invariant under $\hat{D}_x f$ for an orientation-preserving $f \in \mathrm{Diff}^1_{\mathrm{loc}}(\mathbb{R}^d)$. In particular, it is enough to satisfy the covering condition for $\mathcal{W} \subseteq \mathbb{R}^d \times \mathrm{SL}(d,\mathbb{R})$. The next subsection gives a method of doing that.

4.2. **Sufficient conditions for covering: algebraic method.** Following the discussion above, we investigate the covering property for the action of $\mathrm{SL}(d,\mathbb{R})$ on itself.

Recall that the sequence $\{D_i\}_{i=1}^{\infty}$ in $\mathrm{SL}(d,\mathbb{R})$ is quasi-conformal if and only if the set $\{D_n \cdots D_1 : n \in \mathbb{N}\}$ is bounded. Also, the sequence $\{D_i\}_{i=1}^{\infty}$ is κ-conformal, if for any $n \in \mathbb{N}$, $D_n \cdots D_1$ is κ-conformal.

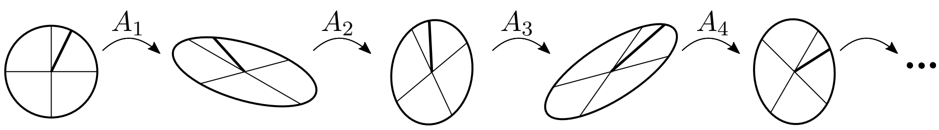


FIGURE 3. A quasi-conformal sequence of matrices

Definition 4.4. For $\kappa \geq 1$ and $\mathcal{D} \subseteq \mathrm{SL}(d,\mathbb{R})$, we say $\langle\mathcal{D}\rangle^+$ has a κ-conformal branch, if there exists a κ-conformal sequence in $\mathcal{D}$. In addition, when $\mathcal{D}$ is finite, we say it robustly has κ-conformal branches, if for every $\tilde{\mathcal{D}}$ in a neighbourhood of $\mathcal{D}$, $\langle\tilde{\mathcal{D}}\rangle^+$ has κ-conformal branches.

For a finite subset $\mathcal{D} \subseteq \mathrm{SL}(d,\mathbb{R})$, if $\overline{\langle\mathcal{D}\rangle^+}$ is not compact, typical branches are not quasi-conformal in a probabilistic sense. More precisely, by assigning positive probabilities to the elements of $\mathcal{D}$, almost every branch with respect to the product measure on $\mathcal{D}^{\mathbb{N}}$ is not κ-conformal for any $\kappa > 1$.

This is worth to mention that for fixed $n \in \mathbb{N}$, for almost every n-tuple $\mathcal{D} \in \mathrm{SL}(d,\mathbb{R})^n$ (w.r.t the natural measure), the Lyapunov spectrum associated to the random product of elements of $\mathcal{D}$ is non-degenerate provided that we assign positive weights to the elements of $\mathcal{D}$. In such case, for almost every branch with respect to the product measure on $\mathcal{D}^{\mathbb{N}}$, the norm of the products diverges exponentially to infinity (cf. [Via14]).

Nevertheless, the following lemma which is an analogue of Theorem 4.1 for the action of $\mathrm{SL}(d,\mathbb{R})$ on itself expresses that the covering condition leads to the existence of bounded branches for a finitely generated semigroup. Moreover, Corollary 4.6 for this special case evidently implies that given $\kappa > 1$, any open neighbourhood $\mathcal{U}$ of the identity in $\mathrm{SL}(d,\mathbb{R})$ with compact closure has a finite subset $\mathcal{D}$ with robustly κ-conformal branches.

Lemma 4.5. *Let $\mathcal{D} \subseteq \mathrm{SL}(d,\mathbb{R})$ be a finite set. Then, there exists $\kappa > 1$ such that $\langle\mathcal{D}\rangle^+$ has a κ-conformal branch if and only if $\overline{\mathcal{U}} \subseteq \mathcal{D}^{-1}\mathcal{U}$ for some $\mathcal{U} \subseteq \mathrm{SL}(d,\mathbb{R})$*

with compact closure. Moreover, if $\mathcal{U}$ is open, $\langle \mathcal{D} \rangle^+$ robustly has a κ-conformal branch for some $\kappa > 1$.

Proof. Let $V = \mathbb{R}^d$ and $\mathcal{W} = \mathbb{R}^d \times \mathcal{U}$. Considering the natural action of $\mathrm{SL}(d, \mathbb{R})$ on $\mathbb{R}^d$, $\mathcal{D}$ can be seen as a family in $\mathrm{Diff}(\mathbb{R}^d)$. Then, the first part of the lemma follows from Theorem 4.1. The second part is similar. Note that $\overline{\mathcal{U}} \subseteq \mathcal{D}^{-1}\mathcal{U}$ implies that for any family $\tilde{\mathcal{D}}$ sufficiently close to $\mathcal{D}$, $\mathcal{U} \subseteq \tilde{\mathcal{D}}^{-1}\mathcal{U}$ holds and so the second conclusion is again a consequence of Theorem 4.1. □

We can deduce the following corollary from Lemma 4.5.

Corollary 4.6. *For any $\kappa > 1$ and any open set $\mathcal{U} \subseteq \mathrm{SL}(d, \mathbb{R})$ containing the identity, there is a finite set $\mathcal{D} \subseteq \mathcal{U}$ such that $\langle \mathcal{D} \rangle^+$ robustly has a κ-conformal branch.*

Proof. Consider small open neighbourhood $\mathcal{V}$ of the identity with compact closure such that every element of $\overline{\mathcal{V}}$ is κ-conformal. Clearly, $\overline{\mathcal{V}} \subseteq \mathcal{U}^{-1}\mathcal{V} = \bigcup_{u \in \mathcal{U}} u^{-1}\mathcal{V}$. By the compactness of $\overline{\mathcal{V}}$, one can choose a finite set $\mathcal{D} \subseteq \mathcal{U}$ with $\overline{\mathcal{V}} \subseteq \mathcal{D}^{-1}\mathcal{V}$. Thus, the conclusion follows from Lemma 4.5. □

Remark 4.7. Lemma 4.5 and Corollary 4.6 can be stated for the existence of bounded branches in an abstract setting for more general topological groups. However, due to the applications for the derivatives of smooth maps, in this paper the discussion is restricted to the special cases of $\mathrm{SL}(d, \mathbb{R})$ and $\mathbb{R}^d$.

Explicit construction for covering. Corollary 4.6 is existential and does not introduce elements of $\mathcal{D}$ explicitly and even does not give any estimate for the cardinality of this set. The next lemma guaranties the covering of small open sets with d^2 elements.

Lemma 4.8. *For any neighbourhood $\mathcal{U}_0$ of the identity in $\mathrm{SL}(d, \mathbb{R})$, there exists an open set $\mathcal{U} \subseteq \mathcal{U}_0$ and a finite set $\mathcal{D} \subseteq \mathcal{U}_0$ with d^2 elements such that $\overline{\mathcal{U}} \subseteq \mathcal{D}^{-1}\mathcal{U}$.*

Denote by $\mathfrak{sl}(d, \mathbb{R})$ the Lie algebra of $\mathrm{SL}(d, \mathbb{R})$ which consists of all $d \times d$ real matrices whose traces are equal to zero. Furthermore, here $\mathfrak{exp}$ denotes the exponential function from a neighbourhood of the zero matrix in $\mathfrak{sl}(d, \mathbb{R})$ to a neighbourhood of the identity in $\mathrm{SL}(d, \mathbb{R})$ which verifies the Baker-Campbell-Hausdorff formula. It will be used in the proof of the next lemma.

Lemma 4.9. *For any neighbourhood $\mathfrak{U}_0$ of the zero matrix in $\mathfrak{sl}(d, \mathbb{R})$, there exist an open subset $\mathfrak{U}$ and a finite subset $\mathfrak{D} = \{w_1, \dots, w_{d^2}\}$ of $\mathfrak{U}_0$ such that $\overline{\mathfrak{exp}(\mathfrak{U})} \subseteq \mathfrak{exp}(\mathfrak{D})^{-1}\mathfrak{exp}(\mathfrak{U})$.*

For the proof, we use the following notation. For a connected open set $U \subseteq \mathbb{R}^N$ and small $t > 0$, denote

$$U_t := \{x \in U : d(x, \partial U) > t\}. \tag{23}$$

In addition, the following observation will be used in the proof of Lemma 4.9.

Lemma 4.10. *Let $U \subseteq \mathbb{R}^N$ be an open set with $\overline{U}$ homeomorphic to the closed unit disk. Suppose that φ_0, φ_1 are two continuous maps defined on a neighbourhood of $\overline{U}$ and are homeomorphisms onto their images. If there exists $t > 0$ such that for any $x \in \overline{U}$, $|\varphi_0(x) - \varphi_1(x)| < t$, then $\big(\varphi_0(U)\big)_t \subseteq \varphi_1(U)$.*

The proof of this lemma is based on considering an affine homotopy between $\varphi_0\big|_{\partial U}$ and $\varphi_1\big|_{\partial U}$, then showing that the image of homotopy does not intersect $\big(\phi_0(U)\big)_t$. Further details are left to the reader.

Proof of Lemma 4.9. Suppose $v_1, \dots, v_N$ are N points in $\mathbb{R}^{N-1}$ with $|v_i| = 1$ such that the origin is contained in their convex hull. Denote the interior of their convex hull by Δ. Clearly, for any small positive $t \in \mathbb{R}$, $\overline{\Delta} \subseteq \bigcup_{i=1}^{N}(\Delta + tv_i)$. Since $\overline{\Delta}$ is compact and $\Delta + tv_j$'s are all open, one can find sufficiently small $c > 0$ such that $\overline{\Delta} \subseteq \bigcup_{i=1}^{N}(\Delta + tv_i)_c$ (following the notation introduced in (23)). As the whole construction is invariant under homothety, for any $r > 0$,

$$r\overline{\Delta} = \overline{r\Delta} \subseteq \bigcup_{i=1}^{N}(r\Delta + trv_i)_{cr}. \tag{24}$$

By identification of $\mathfrak{sl}(d, \mathbb{R})$ with $\mathbb{R}^{N-1}$ for $N = d^2$, v_i's can be seen as $d \times d$ matrices with zero trace. For any $1 \le j \le N$, and any sufficiently small $r > 0$, we define $u_j^{(r)} : r\Delta \to \mathfrak{sl}(d, \mathbb{R})$ as the following

$$u_j^{(r)}(x) = \mathfrak{exp}^{-1}\big(\mathfrak{exp}(trv_j)\mathfrak{exp}(x)\big).$$

By the Baker-Campbell-Hausdorff formula for the Lie groups (see for instance [Ros06]),

$$\mathfrak{exp}^{-1}\big(\mathfrak{exp}(trv_j)\mathfrak{exp}(x)\big) = x + trv_j + \varepsilon_j(x, tr),$$

where ε_j satisfies $|\varepsilon_j(x, s)| < E_j|x|s$ for some $E_j > 0$. Thus, whenever $r < \min_{1\le j\le N}\{\frac{c}{E_j t}\}$, for $1 \le j \le N$ one has,

$$\sup_{x\in\overline{r\Delta}} \big|u_j^{(r)}(x) - (x + trv_j)\big| = \sup_{x\in\overline{r\Delta}} |\varepsilon_j(x, rt)| \le E_j rt \sup_{x\in\overline{r\Delta}} |x| \le E_j r^2 t < cr.$$

By Lemma 4.10, for every $1 \le j \le N$, $r\Delta + trv_j \subseteq u_j^{(r)}(r\Delta)$ and so by (24),

$$\overline{r\Delta} \subseteq \bigcup_{j=1}^{N} u_j^{(r)}(r\Delta).$$

Finally, as the exponential map is a diffeomorphism on $u_j^{(r)}(r\Delta)$ and on $r\Delta$,

$$\overline{\mathfrak{exp}(r\Delta)} = \mathfrak{exp}(\overline{r\Delta}) \subseteq \mathfrak{exp}\Big(\bigcup_{j=1}^{N} u_j^{(r)}(r\Delta)\Big) = \bigcup_{j=1}^{N}\Big(\mathfrak{exp}\big(u_j^{(r)}(r\Delta)\big)\Big). \tag{25}$$

Meanwhile, by the definition of $u_j^{(r)}$, we have $\mathfrak{exp}(u_j^{(r)}(r\Delta)) = \mathfrak{exp}(trv_j)\mathfrak{exp}(r\Delta)$. Thus, the conclusion of Lemma 4.9 follows from (25) by taking $w_j := -trv_j$ and $\mathfrak{U} := r\Delta$ for sufficiently small r. □

Remark 4.11. One can start with any simplex Δ in $\mathfrak{sl}(d, \mathbb{R}) \simeq \mathbb{R}^{d^2-1}$ containing the origin in the interior to provide an explicit formula for $\mathfrak{D}$ in Lemma 4.9.

Proof of Lemma 4.8. Consider an open neighbourhood $\mathfrak{U}_0$ of the zero matrix in $\mathfrak{sl}(d, \mathbb{R})$ such that the $\mathfrak{exp}$ function is a diffeomorphism in a neighbourhood of $\overline{\mathfrak{U}_0}$ and $\exp(\overline{\mathfrak{U}_0}) \subseteq \mathcal{U}_0$. Now, applying Lemma 4.9 to $\mathfrak{U}_0$, one can get $\mathfrak{U}, \mathfrak{D}$. Then, $\mathcal{U} := \mathfrak{exp}(\mathfrak{U})$ and $\mathcal{D} := \mathfrak{exp}(\mathfrak{D})$ satisfy the conditions of Lemma 4.8 and the proof is finished. □

4.3. **Sufficient conditions for covering: analytic method.** This subsection states another approach to derive a sufficient condition leading to the covering property with respect to a bounded subset of $\mathcal{E}(M)$.

Lemma 4.12. *Let $\mathcal{F} \subseteq \mathrm{Diff}^1_{\mathrm{loc}}(M)$ and $V \subseteq M$ be an open set with compact closure. Assume that for any $(x, v) \in T^1M$ with $x \in \overline{V}$, there exists $f \in \mathcal{F}$ satisfying $f(x) \in V$ and $\|\hat{D}_x f|_{v^\perp}\| < 1$. Then, there exists an open set $\mathcal{W} \subseteq \mathcal{E}(M)$ with compact closure such that $\pi(\mathcal{W}) = V$ and $\overline{\mathcal{W}} \subseteq \bigcup_{f\in\mathcal{F}} (\hat{\mathcal{D}}f)^{-1}(\mathcal{W})$.*

Proof. Since the set $\{(x, v) \in T^1M : x \in \overline{V}\}$ is compact, one can find a finite subset $\mathcal{F}_0 \subseteq \mathcal{F}$ and $\epsilon > 0$ such that for any $(x, v) \in T^1M$ with $x \in \overline{V}$, there exists $f \in \mathcal{F}_0$ satisfying $f(x) \in V$ and $\|\hat{D}_x f|_{v^\perp}\| < 1 - \epsilon$.

Denote $\Theta := \max\{\|\hat{D}_x f\| : f \in \mathcal{F}_0, x \in \overline{V}\}$. Let $H \geq \Theta^4/\epsilon$ be a real number. Then, consider $\mathcal{W} \subseteq \mathcal{E}(M)$, defined by

$$\mathcal{W} := \{\mathbf{w} \in \mathcal{E}(M) : \pi(\mathbf{w}) \in V \text{ and } \|\mathbf{w}\|_2 < H\}. \tag{26}$$

Clearly, $\mathcal{W}$ is an open subset with compact closure and $\pi(\mathcal{W}) = V$. It is enough to prove that (22) holds for the subfamily $\mathcal{F}_0$ and $\mathcal{W}$ defined by (26). Consider $\mathbf{w} = (x, (v_1, \dots, v_d)) \in \overline{\mathcal{W}}$. So, $\|\mathbf{w}\|_2 \leq H$. If $\|\mathbf{w}\|_2 < \Theta^{-1}H$ and $f \in \mathcal{F}_0$ with $f(x) \in V$, then by (19),

$$\|\hat{\mathcal{D}}f(\mathbf{w})\|_2 = \big(\sum_{i=1}^{d} |\hat{D}_x f(v_i)|^2\big)^{\frac{1}{2}} \leq \|\hat{D}_x f\|.\|\mathbf{w}\|_2 < H.$$

Thus, $\hat{\mathcal{D}}f(\mathbf{w}) \in \mathcal{W}$ and consequently, $\mathbf{w} \in \bigcup_{f\in\mathcal{F}_0} (\hat{\mathcal{D}}f)^{-1}(\mathcal{W})$. Now, assume that $\|\mathbf{w}\|_2 \in [\Theta^{-1}H, H]$. Without loss of generality, assume that $|v_1| \geq \cdots \geq |v_d|$. Clearly, $|v_d| \leq \sqrt{H/d}$. So,

$$|v_d|\Theta\sqrt{d/\epsilon} \leq \Theta\sqrt{H/\epsilon} \leq \Theta^{-1}H \leq \|\mathbf{w}\|_2 = \sqrt{|v_1|^2 + \cdots + |v_d|^2} \leq \sqrt{d}|v_1|.$$

This implies that $\Theta^2|v_d|^2 \leq \epsilon|v_1|^2$. Let v be a vector perpendicular to $v_1, \dots, v_{d-1}$. By assumption, one can choose $f \in \mathcal{F}_0$ with $f(x) \in V$ and $\|\hat{D}_x f|_{v^\perp}\| < 1 - \epsilon$. Then,

$$\begin{aligned}
\|\hat{\mathcal{D}}f(\mathbf{w})\|_2^2 &= \sum_{i=1}^{d} |\hat{D}_x f(v_i)|^2 \\
&< (1-\epsilon)^2(|v_1|^2 + \cdots + |v_{d-1}|^2) + \Theta^2|v_d|^2 \\
&\leq (1-\epsilon)^2(|v_1|^2 + \cdots + |v_{d-1}|^2) + \epsilon|v_1|^2 < \|\mathbf{w}\|_2^2 \leq H^2.
\end{aligned}$$

Therefore, $\mathbf{w} \in \bigcup_{f\in\mathcal{F}_0} (\hat{\mathcal{D}}f)^{-1}(\mathcal{W})$. □

5. Quasi-conformal blenders

This section is devoted to a new mechanism/phenomenon that we call quasi-conformal blender. For pseudo-semigroup actions, the quasi-conformal blender guarantees the existence of quasi-conformal expanding orbit-branches at every point in some region which leads to the stable local ergodicity. Here, we present the proof of Theorem F and its variants using the results of the previous sections.

Throughout the section, M is a boundaryless, not necessarily compact, smooth Riemannian manifold of dimension d. Recall that $x \in M$ is a *(Lebesgue) density* *point* of a measurable set $S \subseteq M$ if

$$\lim_{r\to 0} \frac{\mathrm{Leb}\big(S\cap B(x,r)\big)}{\mathrm{Leb}\big(B(x,r)\big)} = 1.$$

It is well-known that a measurable set $S \subseteq M$ is equal to the set of its Lebesgue density points, up to a zero measure set. Moreover, for $f \in \mathrm{Diff}^1_{\mathrm{loc}}(M)$, $f(x)$ is a density point of $f(S)$, provided that $x \in \mathrm{Dom}(f)$ is a density point of S.

Definition 5.1 (ρ-ergodic)**.** Let $V \subseteq M$ be an open set with compact closure. For $\rho > 0$ and $\mathcal{F} \subseteq \mathrm{Diff}^1_{\mathrm{loc}}(M)$, we say $\mathrm{IFS}(\mathcal{F}{\downarrow_V})$ is *ρ-ergodic* (w.r.t. Leb.), if the set of density points of every measurable $\mathcal{F}{\downarrow_V}$-invariant subset of V either is empty or contains a ball of radius ρ.

Clearly, this definition is equivalent to say that every measurable $\mathcal{F}{\downarrow_V}$-invariant set of positive measure in V contains a ball of radius ρ, up to a set of zero Lebesgue measure.

5.1. **From quasi-conformal expansion to local ergodicity.** In this subsection, we state a technical lemma about the quasi-conformal expanding sequences. It will be used in both local and global settings for proving local ergodicity.

Lemma 5.2 (Local ergodicity)**.** *Let $\eta, \kappa > 1$ and $\rho, \alpha, C > 0$. Also, let $\{x_j\}_{j=0}^{\infty}$ be a sequence in M and $\{f_j\}_{j=1}^{\infty}$ with $f_j : B(x_{j-1},\rho) \to f_j(B(x_{j-1},\rho))$ be a sequence in $\mathrm{Diff}^{1+\alpha}_{\mathrm{loc}}(M)$ satisfying $m(Df_j) > \eta$ and $f_j(x_{j-1}) = x_j$. Then,*

- (a) *for any open neighbourhood $U \subseteq B(x_0,\rho)$ of x_0, there exists $n \in \mathbb{N}$ with $B(x_n,\rho) \subseteq f^n(U)$.*
- (b) *If in addition, the set $\{x_0, x_1, \ldots\}$ is bounded, the sequence $\{f_j\}_{j=1}^{\infty}$ is κ-conformal at x_0, and $\|Df_j\|_{\mathcal{C}^{1+\alpha}} < C$, then for any measurable set $S \subseteq B(x_0,\rho)$ with density point at x_0, there exists $n \in \mathbb{N}$ such that $\bigcup_{i\in\mathbb{N}} f^i(S)$ contains $B(x_n,\rho)$, up to a set of zero Lebesgue measure.*

Proof. We prove assertions (a) and (b) separately.

Proof of (a). For every $i \geq 1$, let s_i be the largest positive number in $(0,\rho]$ satisfying $B(x_i,s_i) \subseteq f^i(U)$. Since $m(Df_i|_{B(x_{i-1},\rho)}) > \eta$, if for some $i \geq 1$, $s_i < \rho\eta^{-1}$, then $s_{i+1} > \eta s_i$, and if $s_i \geq \rho\eta^{-1}$, then $s_{i+1} = \rho$. Hence, there is $n \geq 0$ with $s_n = \rho$. This finishes the proof of part (a). Note that for this part of the lemma, we only need the sequence $\{f_i\}$ to be $\mathcal{C}^1$-regular.

Proof of (b). By applying Theorem 3.1 to this sequence, one can find $\xi_0 > 0$ and $\theta > 1$ such that for each $j \in \mathbb{N}$, there is $r_j > 0$ with

$$f^j(B(x_0,r_j)) \subseteq B(x_j,\xi_0) \subseteq f^j(B(x_0,\theta r_j)), \tag{27}$$

and $\lim\limits_{j\to\infty} r_j = 0$. For the rest of the proof, fix ξ_0 and assume that $\xi_0 < \rho$. Denote $\hat{S} := \bigcup_{i\geq 0} f^i(S)$. Since x_0 is a density point of S, for any $\epsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that whenever $j > j_0$,

$$\frac{\mathrm{Leb}\big(B(x_0,\theta r_j)\setminus \hat{S}\big)}{\mathrm{Leb}\big(B(x_0,\theta r_j)\big)} < \epsilon.$$

There exists $\sigma > 0$ such that for any $r, r' \in (0, \rho)$,

$$\frac{\mathrm{Leb}(B(x_0, r))}{\mathrm{Leb}(B(x_0, r'))} \le \sigma(\frac{r}{r'})^d.$$

Denote $f^{-j} := f_1^{-1} \circ \cdots \circ f_j^{-1}$. Then, by (27),

$$\begin{aligned}\frac{\mathrm{Leb}\big(f^{-j}\big(B(x_j,\xi_0)\setminus\hat{S}\big)\big)}{\mathrm{Leb}\big(f^{-j}(B(x_j,\xi_0))\big)} &\le \frac{\mathrm{Leb}\big(B(x_0,\theta r_j)\setminus\hat{S}\big)}{\mathrm{Leb}\big(B(x_0,\theta r_j)\big)}\cdot\frac{\mathrm{Leb}\big(B(x_0,\theta r_j)\big)}{\mathrm{Leb}\big(f^{-j}(B(x_j,\xi_0))\big)}\\ &\le \frac{\mathrm{Leb}\big(B(x_0,\theta r_j)\setminus\hat{S}\big)}{\mathrm{Leb}\big(B(x_0,\theta r_j)\big)}\cdot\frac{\mathrm{Leb}\big(B(x_0,\theta r_j)\big)}{\mathrm{Leb}\big(B(x_0,r_j)\big)}\\ &< \epsilon\sigma\theta^d.\end{aligned}$$

It follows from Lemma 3.4 that for some $L > 1$,

$$\frac{\mathrm{Leb}\big(B(x_j,\xi_0)\setminus\hat{S}\big)}{\mathrm{Leb}\big(B(x_j,\xi_0)\big)} < L\frac{\mathrm{Leb}\big(f^{-j}(B(x_j,\xi_0)\setminus\hat{S})\big)}{\mathrm{Leb}\big(f^{-j}(B(x_j,\xi_0))\big)} < \epsilon\sigma L\theta^d.$$

Thus, for any $j > j_0$,

$$\frac{\mathrm{Leb}\big(\hat{S}\cap B(x_j,\xi_0)\big)}{\mathrm{Leb}\big(B(x_j,\xi_0)\big)} > 1 - \epsilon\sigma L\theta^d. \tag{28}$$

Now, since ϵ was arbitrary, by (28), the density of $\hat{S}$ in $B(x_j, \xi_0)$ tends to 1 (as $j \to \infty$). Then, for each accumulation point y_0 of the bounded sequence $\{x_i\}_{i=1}^{\infty}$, $\hat{S}$ contains $B(y_0, \xi_0)$ up to a set of zero Lebesgue measure. Next, take a sufficiently large i such that $x_i \in B(y_0, \xi_0)$. This implies that $\hat{S}$ contains an open neighbourhood U of x_i, up to a set of zero Lebesgue measure. Then, by part (a), there exists $n > i$ such that $B(x_n, \rho) \subseteq f^{n-i}(U)$. Finally, since diffeomorphisms maps sets of zero Lebesgue measure to sets of zero Lebesgue measure, $\hat{S}$ contains $B(x_n, \rho)$, up to a set of zero Lebesgue measure. □

Lemma 5.2 has the following global consequence which can be seen as a generalization of Theorem 1.1.

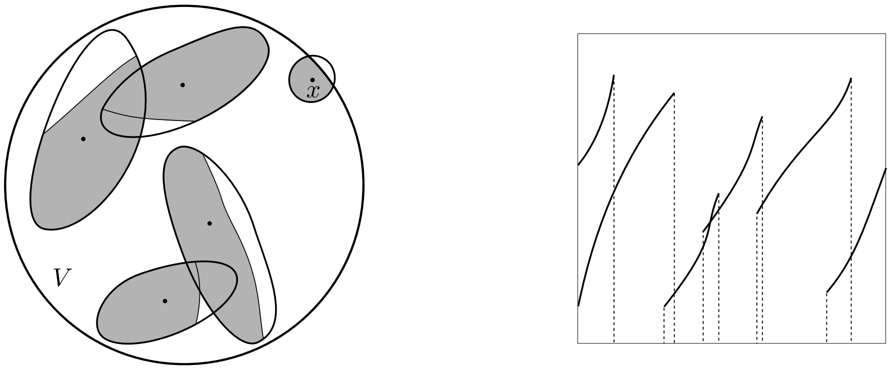
Theorem 5.3. *Let M be a closed manifold and $\mathcal{F} \subseteq \mathrm{Diff}^{1+\alpha}(M)$ be finite. Suppose that there exist $\eta, \kappa > 1$ such that $\mathrm{IFS}(\mathcal{F})$ has a κ-conformal η-expanding orbit-branch at every point. Then, $\mathrm{IFS}(\mathcal{F})$ is ρ-ergodic for some $\rho > 0$. In particular, the action of a group $G \subseteq \mathrm{Diff}^1(M)$ is ergodic, if it is minimal and $\mathcal{F} \subseteq G$.*

Proof. Since $\mathcal{F}$ is finite, there exist $\rho > 0$ and $\eta' \in (1, \eta)$ such that whenever $m(D_x f) > \eta$, for some $x \in M$ and $f \in \mathcal{F}$, then $m(Df|_{B(x,\rho)}) > \eta'$. Indeed, if $C := \max_{f\in\mathcal{F}} \|f\|_{C^{1+\alpha}}$, then for any $x, y \in M$,

$$|m(D_x f) - m(D_y f)| \le Cd(x, y)^{\alpha}.$$

where $d(.,.)$ denotes the distance on M. So, $m(D_x f) > \eta$ implies that $m(D_y f) > \eta'$, provided that $d(x, y)^{\alpha} < \rho := C^{-1}(\eta - \eta')$.

For $f \in \mathcal{F}$, denote $U_f := \{x \in M : m(D_x f) > \eta'\}$ and $\hat{f} := f|_{U_f}$. Also, let $\hat{\mathcal{F}} := \{\hat{f} : f \in \mathcal{F}\}$. Consider a measurable $\mathcal{F}$-invariant subset S of positive measure and pick x to be a density point of S. Then, the κ-conformal η-expanding orbit-branch of $\mathrm{IFS}(\hat{\mathcal{F}})$ at x, provides a sequence of maps satisfying the assumptions of Lemma 5.2 and the conclusion follows from this lemma.



Images of a neighbourhood of x One-dimensional localized maps

FIGURE 4. Families of expanding maps satisfying the covering condition in Theorem F.

For the second part, denote the set of density points of S by $S^\bullet$. Since $\mathcal{F} \subseteq G$, it follows from the first part that $S^\bullet$ contains an open ball B. We claim that $S^\bullet = M$. Indeed, $\mathrm{Leb}(B \setminus S) = 0$ implies that for every $g \in G$, $\mathrm{Leb}(g(B) \setminus g(S)) = 0$. Then, by the invariance of S, $\mathrm{Leb}(g(B) \setminus S) = 0$ and in particular, $g(B) \subseteq S^\bullet$. On the other hand, by the minimality assumption, $\bigcup_{g\in G} g(B) = M$. This proves the claim and finishes the proof of the theorem. □

5.2. **Proof of Theorem F.** We will prove the following theorem which in particular implies Theorem F.

Theorem 5.4 (Quasi-conformal blender). *Let $\mathcal{F} \subseteq \mathrm{Diff}^{1+}_{\mathrm{loc}}(M)$. Let $\mathcal{W} \subseteq \mathcal{E}(M)$ be an open set with compact closure and $V := \pi(\mathcal{W})$. Assume that for any $\mathbf{w} \in \overline{\mathcal{W}}$, there exists $f \in \mathcal{F}$ satisfying*

(i) *$\hat{\mathcal{D}}f(\mathbf{w}) \in \mathcal{W}$,*
(ii) *$m(D_x f) > 1$, where $x = \pi(\mathbf{w})$.*

Then, there exist real numbers $\rho > 0$ and $\kappa > 1$ such that for every $\tilde{\mathcal{F}} \subseteq \mathrm{Diff}^{1+}_{\mathrm{loc}}(M)$ in a $\mathcal{C}^1$-neighbourhood of $\mathcal{F}$,

(a) *for any $x \in V$, $\mathrm{IFS}(\tilde{\mathcal{F}}{\downarrow_V})$ has an orbit-branch which is κ-conformal at x,*
(b) *$\mathrm{IFS}(\tilde{\mathcal{F}}{\downarrow_V})$ is ρ-ergodic.*

In addition, if M is compact, $\mathrm{IFS}(\mathcal{F})$ and $\mathrm{IFS}(\mathcal{F}^{-1})$ are minimal, then $\mathrm{IFS}(\mathcal{F})$ is $\mathcal{C}^1$-stably ergodic in $\mathrm{Diff}^{1+}_{\mathrm{loc}}(M)$ and $\mathrm{IFS}(\mathcal{F}^{-1})$ is $\mathcal{C}^1$-robustly minimal.

Proof of Theorem F. It follows from (1) that for every $\mathbf{w} \in \mathcal{W}$, there exists $f \in \mathcal{F}$ with $\hat{\mathcal{D}}f(\mathbf{w}) \in \mathcal{W}$. Now, since every element of $\mathcal{F}$ is expanding, one has $m(D_x f) > 1$ for $x = \pi(\mathbf{w})$. So, Theorem F follows from part (b) of Theorem 5.4. □

Remark 5.5. If a family $\mathcal{F}$ and a set $\mathcal{W}$ satisfy the assumptions of Theorem 5.4, then for $\mathbf{w} \in \overline{\mathcal{W}}$, one can get a subfamily $\mathcal{F}_{\mathbf{w}} \subseteq \mathcal{F}$ consisting of all elements satisfying (i)-(ii). Then, for any element $f \in \mathcal{F}_{\mathbf{w}}$, restrict its domain to a small neighborhood of $\mathbf{w}$ such that the restricted map is expanding. Let $\mathcal{F}'$ be the family of all these restricted diffeomorphisms. It is clear that $\mathcal{F}'$ and $\mathcal{W}$ satisfy the assumption of Theorem F. In other words, one can deduce part (b) in Theorem 5.4 from Theorem F, and vice-versa.

Remark 5.6. As a matter of fact, the $\mathcal{C}^1$ stability in Theorem 5.4 and its consequences in this paper are valid in a substantially stronger form that allows to perturb the family at each step of iterations. We do not discuss the details in this paper. Cf. [HN14], where this notion of *strong stability* has been introduced.

Proof of Theorem 5.4. Since the assumptions (i)-(ii) are stable under small perturbation of f, in the $\mathcal{C}^1$ topology, and $\mathbf{w} \in \mathcal{E}(M)$, one can deduce that

- For every $\mathbf{w} \in \overline{\mathcal{W}}$ there exist $\epsilon(\mathbf{w}) > 0$ and $f_{\mathbf{w}} \in \mathcal{F}$ such that if $\mathcal{B}_{\mathbf{w}}$ denotes the open ball of radius $\epsilon(\mathbf{w})$ with center $\mathbf{w} \in \mathcal{E}(M)$, then for any $\tilde{f}$ sufficiently close to $f_{\mathbf{w}}$, in the $\mathcal{C}^1$ topology, $\pi(\mathcal{B}_{\mathbf{w}}) \subseteq \mathrm{Dom}(\tilde{f})$ and for any $\mathbf{w}' \in \mathcal{B}_{\mathbf{w}}$,
  (1) $\hat{\mathcal{D}}\tilde{f}(\mathbf{w}') \in \mathcal{W}$,
  (2) $m(D_{x'}\tilde{f}) > 1$, where $\pi(\mathbf{w}') = x'$.

Proof of (a). Let $\mathcal{B}'_{\mathbf{w}}$ be the open ball of radius of $\frac{1}{2}\epsilon(\mathbf{w})$ with center $\mathbf{w}$. Then, by the compactness of $\overline{\mathcal{W}}$, there is a finite subset $\{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ of $\overline{\mathcal{W}}$ such that $\overline{\mathcal{W}} \subseteq \bigcup_i \mathcal{B}'_{\mathbf{w}_i}$. By Corollary 4.2, this implies the existence of $\kappa > 1$ such that for every $\tilde{\mathcal{F}}$ sufficiently close to $\mathcal{F}$, in the $\mathcal{C}^1$ topology, IFS$(\tilde{\mathcal{F}}{\downarrow_V})$ has a κ-conformal orbit-branch at every point of V. So, the proof of part (a) is finished.

Proof of (b). Let $\rho > 0$ be smaller than the Lebesgue number of the open covering $\bigcup_i \mathcal{B}'_{\mathbf{w}_i}$ for $\overline{\mathcal{W}}$. Then, there is $\eta > 1$ such that for any $\mathbf{w} \in \overline{\mathcal{W}}$, there exists $1 \leq i \leq k$ such that for any $\tilde{f}$ sufficiently close to $f_{\mathbf{w}_i}$, in the $\mathcal{C}^1$ topology, $B(x, \rho) \subseteq \mathrm{Dom}(\tilde{f})$, $\tilde{f}(B(x, \rho)) \subseteq V$, and $m(D\tilde{f}|_{B(x,\rho)}) > \eta$, where $x = \pi(\mathbf{w})$.

Denote $\tilde{\mathcal{F}}_0 := \{\tilde{f}_{\mathbf{w}_1}, \dots, \tilde{f}_{\mathbf{w}_k}\}$. Let $\alpha > 0$ be such that $\tilde{\mathcal{F}}_0 \subseteq \mathrm{Diff}^{1+\alpha}_{\mathrm{loc}}(M)$. Consider a measurable $\tilde{\mathcal{F}}_0{\downarrow_V}$-invariant set S of positive measure and let x_0 be a density point of S. Family $\mathcal{F}_0$ satisfies the assumptions (i)-(ii) of the theorem, so it follows from part (a) that IFS$(\tilde{\mathcal{F}}_0|_V)$ has a κ-conformal orbit-branch $\{x_i\}_{i=0}^{\infty}$ at x_0. Let $\{\tilde{f}_i\}_{i=1}^{\infty}$ be the sequence of maps providing this orbit-branch, namely $f_i(x_{i-1}) = x_i$ for every $i \in \mathbb{N}$. Next, our aim is to apply Lemma 5.2 to this sequence. The problem is that x_0 may be close to the boundary of V and $B(x_0, \rho) \not\subseteq \tilde{f}_1{\downarrow_V}$. To avoid this challenge, we remove the first term of the sequences and consider $\{\tilde{f}_i\}_{i=2}^{\infty}$ and $\{x_i\}_{i=1}^{\infty}$, which by means of above arguments satisfy the assumptions of Lemma 5.2. Note that $x_1 = \tilde{f}_1(x_0)$ is also a density point of the invariant set S. By part (b) of Lemma 5.2, there is $\xi_0 > 0$ and an accumulation point y_0 of $\{x_i\}_{i=1}^{\infty}$ such that S contains $B(y_0, \xi_0)$, up to a set of measure zero. Since for any $i \geq 1$, $B(x_i, \rho) \subseteq V$ and so $B(y_0, \rho) \subseteq V$. Similarly, an expanding orbit-branch $\{y_i\}_{i=0}^{\infty}$ of IFS$(\tilde{\mathcal{F}}{\downarrow_V})$ at y_0 can be provided in such a way that for any $i \geq 0$, $B(y_i, \rho) \subseteq V$. Using part (a) of Lemma 5.2 for open set $U = B(y_0, \xi_0)$, one can conclude that the set of density points of S contains $B(y_n, \rho)$ for some $n \geq 0$. This finishes the proof of part (b).

Next, we go to the proof of global results. We assume that M is compact, IFS$(\mathcal{F})$ and IFS$(\mathcal{F}^{-1})$ are minimal on M. By the minimality, $\langle \mathcal{F} \rangle^+(B(x, \rho')) = \langle \mathcal{F}^{-1} \rangle^+(B(x, \rho')) = M$ where $\rho' < \frac{1}{2}\rho$. Suppose that $\bigcup_{x \in X} B(x, \rho') = M$ for some finite set $X \in M$. On the other hand, by the compactness of M, there exists a finite set $\mathcal{F}_1 \subseteq \langle \mathcal{F} \rangle^+$ with $\mathcal{F}_1(B(x, \rho')) = \mathcal{F}_1^{-1}(B(x, \rho')) = M$, for any $x \in X$. Consider small perturbation $\tilde{\mathcal{F}}$ of $\mathcal{F}$ and denote the elements of $\tilde{\mathcal{F}}$ corresponding to the family $\mathcal{F}_1$ by $\tilde{\mathcal{F}}_1$. If the perturbation is sufficiently small, then for every ball B of radius ρ,

$$\tilde{\mathcal{F}}_1(B) = \tilde{\mathcal{F}}_1^{-1}(B) = M. \tag{29}$$

Proof of stable ergodicity. Let S be a measurable $\tilde{\mathcal{F}}$-invariant subset of M with positive measure. Consider an arbitrary ball $B_0 \subseteq V$ of radius ρ. By (29), $S \subseteq \tilde{\mathcal{F}}_1^{-1}(B_0) = M$ and so $\mathrm{Leb}(S \cap B_0) > 0$. Then, by ρ-ergodicity of $\mathrm{IFS}(\tilde{\mathcal{F}}{\downarrow_V})$, the set of density points of S contains some ball $B_1 \subseteq M$ of radius ρ. Finally, by (29), $\tilde{\mathcal{F}}_1(B_1) = M$ and this implies that S contains M, up to a set of measure zero.

Proof of robust minimality. The proof of robust minimality is similar to the one of stable ergodicity. Consider an open set $U \subseteq M$, and ball B_0 as above. Let $\tilde{\mathcal{F}} \subseteq \mathrm{Diff}^1_{\mathrm{loc}}(M)$ be a family in a small $\mathcal{C}^1$-neighbourhood of $\mathcal{F}$ satisfying (29). Again, (29) implies that $\hat{U} := \langle \tilde{\mathcal{F}} \rangle^+(U)$ intersects $B_0 \subseteq V$. Consider $z_0 \in \hat{U} \cap B_0$. Then, similar to the arguments of part (b), one can find an expanding orbit-branch of $\mathrm{IFS}(\tilde{\mathcal{F}}{\downarrow_V})$ at z_0 and use part (a) of Lemma 5.2 to deduce that $\hat{U}$ contains a ball $B_2 \subseteq V$ of radius ρ. Finally, again by (29), $M = \tilde{\mathcal{F}}_1(B_2) \subseteq \hat{U}$. Since open set U was arbitrarily chosen, it follows that the orbit of every point under $\mathrm{IFS}(\tilde{\mathcal{F}}^{-1})$ is dense in M. This means $\mathrm{IFS}(\tilde{\mathcal{F}}^{-1})$ is minimal. □

5.3. **Contracting quasi-conformal blenders.** In this subsection, we present a variant of Theorem F in the setting of contracting maps. This is useful to get local ergodicity and local minimality simultaneously.

Theorem 5.7 (Contracting quasi-conformal blender)**.** *Let $\mathcal{G} \subseteq \mathrm{Diff}^{1+}_{\mathrm{loc}}(M)$ and $\mathcal{W} \subseteq \mathcal{E}(M)$ be a bounded open set with $V := \pi(\mathcal{W})$. Let $U \subseteq M$ be an open set with compact closure containing $\pi(\overline{\mathcal{W}})$. Also, assume that*

(i) $\overline{\mathcal{W}} \subseteq \bigcup_{g \in \mathcal{G}} \hat{\mathcal{D}}g(\mathcal{W})$,
(ii) *for every $g \in \mathcal{G}$, $\overline{U} \subseteq \mathrm{Dom}(g)$ and $g(\overline{U}) \subseteq U$.*
(iii) *every $g \in \mathcal{G}$ is uniformly contracting on U.*

Then, for every $\tilde{\mathcal{G}} \subseteq \mathrm{Diff}^{1+}_{\mathrm{loc}}(M)$ in a $\mathcal{C}^1$-neighbourhood of $\mathcal{G}$, any measurable $\tilde{\mathcal{G}}{\downarrow_U}$-invariant set of positive measure in U, has full measure in V. Also, for every $x \in U$, $\langle \mathcal{G}{\downarrow_U} \rangle^+(x)$ is dense in V.

Proof. By the compactness of $\overline{\mathcal{W}}$ and the openness of $\mathcal{W}$, one can replace $\mathcal{G}$ with a finite subfamily satisfying the assumptions. So, we may assume that $\mathcal{G} \subseteq \mathrm{Diff}^{1+\alpha}_{\mathrm{loc}}(M)$, for some $\alpha > 0$, is finite.

Let $\Lambda := \cap_n \mathcal{G}^n(\overline{U})$ be the Hutchinson attractor of $\mathrm{IFS}(\mathcal{G}|_U)$. It follows from [Hut81] that for any $x \in U$, $\langle \mathcal{G} \rangle^+(x)$ is dense in Λ (cf. [HN14, Theorem 4.2]). On the other hand, for any $n \in \mathbb{N}$,

$$V \subseteq \mathcal{G}(V) \subseteq \mathcal{G}^n(V) \subseteq \mathcal{G}^n(\overline{U}),$$

and thus $\overline{V} \subseteq \Lambda$. This in particular implies that for every $x \in U$, $\langle \mathcal{G}{\downarrow_U} \rangle^+(x)$ is dense in V and consequently, every $\mathcal{G}{\downarrow_U}$-invariant subset of positive measure in U, intersects V in a set of positive measure.

Note that the family $\mathcal{G}^{-1}{\downarrow_U}$ is uniformly expanding on V. On the other hand, in view of assumption (i), $\mathcal{G}^{-1}$ satisfies the covering property (1) for $\mathcal{W}$. Now, let S be a measurable $\mathcal{G}{\downarrow_U}$-invariant set with $\mathrm{Leb}(S) > 0$. It follows from above that $\mathrm{Leb}(S \cap V) > 0$. Suppose that $\mathrm{Leb}(S \cap V) < \mathrm{Leb}(V)$. Then, $S' := V \setminus S$ is $\mathcal{G}^{-1}{\downarrow_V}$-invariant and $\mathrm{Leb}(S') > 0$. By applying Theorem 5.4 to the family $\mathcal{G}^{-1}{\downarrow_V}$, one gets that the set of density points of S' contains an open ball B. Let x be a density point of S. By above arguments, some element of $\langle \mathcal{G}{\downarrow_U} \rangle^+$ maps x to B. This contradicts to the invariance of S under $\mathcal{G}{\downarrow_U}$ and shows that S must have full measure in V.

Finally, note that after small perturbation of the generators of $\mathcal{G}$, all the assumptions of the theorem hold. That is, the conclusion follows for the family $\tilde{\mathcal{G}}$ sufficiently close to $\mathcal{G}$, following similar arguments above. □

Remark 5.8. It follows from the proof of Theorem 5.7 that one can replace the assumption (i) with $\overline{V} \subseteq \bigcup_{g\in\mathcal{G}} g(V)$ and deduce the density of $\langle\mathcal{G}\rangle^+(x)$ in V for any $x \in V$.

5.4. **Other statements on quasi-conformal blenders.** In this subsection, we state an analogue of Theorem F. First note that if $V \subseteq M$ is completely inside an open chart of M, one can use the coordinating map to translate the problem to an open subset of the Euclidean space. In this case, one can get the following statement in view of Theorem 5.4.

Theorem 5.9. *Let $\mathcal{F} \subseteq \mathrm{Diff}^{1+}_{\mathrm{loc}}(\mathbb{R}^d)$ be a family of orientation-preserving maps. Let $V \subseteq \mathbb{R}^d$ be a bounded open set and $\mathcal{U} \subseteq \mathrm{SL}(d,\mathbb{R})$ be a bounded open neighbourhood of the identity. Assume that for any $x \in \overline{V}$, there exists $\mathcal{F}_x \subseteq \mathcal{F}$ such that*

(i) *$f(x) \in V$, for $f \in \mathcal{F}_x$,*
(ii) *$\overline{\mathcal{U}} \subseteq \bigcup_{f\in\mathcal{F}_x} (\hat{D}_x f)^{-1}\mathcal{U}$,*
(iii) *$m(D_x f) > 1$, for $f \in \mathcal{F}_x$.*

Then, there exists $\rho > 0$ such that for every $\tilde{\mathcal{F}} \subseteq \mathrm{Diff}^{1+}_{\mathrm{loc}}(M)$ in a $\mathcal{C}^1$-neighbourhood of $\mathcal{F}$, $\mathrm{IFS}(\tilde{\mathcal{F}}{\downarrow_V})$ is ρ-ergodic.

Note that V is not necessarily connected in Theorem 5.9. This yields a flexible tool to get a result similar to Theorem F for every relatively compact open set in manifolds using local charts.

Let M be a boundaryless manifold of dimension d with a finite atlas of charts $\{(W_i, h_i)\}_{i\in I}$ on M such that for any $i \in I$, $h_i : W_i \to h_i(W_i)$ is a $\mathcal{C}^2$ diffeomorphism and $h_i(W_i)$ is an open disk in $\mathbb{R}^d$. Assume that $h_i(W_i) \cap h_j(W_j) = \emptyset$ if $i \neq j$. For $i \in I$, let W_i' be an open set such that $\overline{W_i'} \subseteq W_i$ and $M = \bigcup_{i\in I} W_i'$. Now, for an open set $V \subseteq M$ with compact closure, denote $V^* := \bigcup_{i\in I} h_i(V \cap W_i')$. Note that $V^* \subseteq \mathbb{R}^d$ is an open set with compact closure. Let $\mathcal{F} \subseteq \mathrm{Diff}^{1+}_{\mathrm{loc}}(M)$ and denote $\mathcal{F}^* \subseteq \mathrm{Diff}^{1+}_{\mathrm{loc}}(\mathbb{R}^d)$,

$$\mathcal{F}^* := \{h_j \circ f \circ h_i^{-1} : f \in \mathcal{F} \cup \{\mathrm{Id}\},\ i,j \in I \text{ such that } x \in W_i \text{ and } f(x) \in W_j\}.$$

It is easy to see that the dynamical properties of $\mathcal{F}$ are translated to the ones of $\mathcal{F}^*$, and vice-versa. Then, we get the following.

Theorem 5.10. *Let $\mathcal{F}, \mathcal{F}^*, V, V^*$ be as above. Let also $\mathcal{U} \subseteq \mathrm{SL}(d,\mathbb{R})$. If $\mathcal{F}^*, V^*, \mathcal{U}$ satisfies the assumptions of Theorem 5.9, then there exists $\rho > 0$ such that for every $\tilde{\mathcal{F}} \subseteq \mathrm{Diff}^{1+}_{\mathrm{loc}}(M)$ in a $\mathcal{C}^1$-neighbourhood of $\mathcal{F}$, $\mathrm{IFS}(\tilde{\mathcal{F}}{\downarrow_V})$ is ρ-ergodic.*

In addition, if M is compact, and the action of $\langle\mathcal{F}\rangle$ is minimal, then it is $\mathcal{C}^1$-stably ergodic and $\mathcal{C}^1$-robustly minimal.

Proof. The first part is an immediate consequence of Theorem 5.9, since the smooth maps send the sets of zero Lebesgue measure to the sets of zero Lebesgue measure. Moreover, Theorem 5.9 shows that for every family $\tilde{\mathcal{F}}$ in a $\mathcal{C}^1$-neighbourhood of $\mathcal{F}$, $\mathrm{IFS}(\tilde{\mathcal{F}}{\downarrow_V})$ is ρ-ergodic.

The second part is a duplication of the last parts of Theorem 5.4, in which minimality implies the stable covering of M by the images of arbitrary balls of radius ρ in V, under finitely many elements of $\langle\mathcal{F}\rangle$. □

6. Stably ergodic actions on manifolds

In this section, we prove Theorems A, B and C using the local results of the previous sections.

6.1. **Ergodic IFS on arbitrary manifold.** In this subsection, we use the results of the previous section to construct a pair of diffeomorphisms generating a $\mathcal{C}^1$-stably ergodic, $\mathcal{C}^1$-robustly minimal IFS on any closed manifold M of dimension d. Theorem B is a consequence of the following.

Theorem 6.1. *Every closed manifold M admits a semigroup generated by two smooth diffeomorphisms that acts $\mathcal{C}^1$-stably ergodic and $\mathcal{C}^1$-robustly minimal in $\mathrm{Diff}^s(M)$, $s \in (1, \infty]$.*

Remark 6.2. As far as the authors know, this is the first example of a stably ergodic action on a manifold of dimension greater than one. In [BFMS17] and [Sar15] the existence of such actions on surfaces is claimed, however, with an argument that only works for conformal actions and mistakenly assumes "conformality of maps with complex eigenvalues". Indeed, it is easy to provide a non-quasiconformal sequence of matrices all with complex eigenvalues.

We will use the following lemma. We omit its proof, since it is very similar to the last part of the proof of Theorem 5.4.

Lemma 6.3. *Let V be an open subset of compact manifold M and $\mathcal{F}$ be a family in $\mathrm{Diff}^1(M)$. Assume that $\langle\mathcal{F}\rangle^+(V) = \langle\mathcal{F}^{-1}\rangle^+(V) = M$. If every measurable $\mathcal{F}$-invariant set has either zero or full Lebesgue measure in V, then $\mathrm{IFS}(\mathcal{F})$ is ergodic on M. Similarly, if for every $x \in M$, $\langle\mathcal{F}\rangle^+(x)$ is dense in V, then $\mathrm{IFS}(\mathcal{F})$ is minimal.*

Proof of Theorem 6.1. The case $\dim(M) = 1$ is known. It comes from Theorem 1.1 and [GI00]. Indeed, Theorem 6.5 provides an explicit example for this case.

So, we may assume that $\dim(M) \geq 2$. The same statement for robust minimality (without the ergodicity) is proved in [HN14, Theorem A]. Here, we carefully modify its proof and make use of our results in the previous sections to prove both robust minimality and stable ergodicity. First, we establish some local construction on the Euclidean space and then realize them by $\mathcal{C}^\infty$ diffeomorphisms on an arbitrary smooth manifold. Recall that we work with the $\mathcal{C}^1$ topology on $\mathrm{Diff}^s(M)$.

Step 1. *Local construction: finitely many generators on $\mathbb{R}^d$.*

In this step, we construct a family of contracting affine transformations in $\mathbb{R}^d$ sufficiently close to Id and satisfying the assumptions of Theorem 5.7.

Fix $\epsilon > 0$ to be sufficiently small number and let $\kappa < 1 + \epsilon$. Consider open neighbourhood $\mathcal{U}_0$ of the identity in $\mathrm{SL}(d, \mathbb{R})$ such that any element $D \in \mathcal{U}_0$ is κ-conformal. Consequently, $1 - \epsilon < m(D)$ and $\max\{\|D\|, \|D^{-1}\|\} < 1 + \epsilon$. By Lemma 4.8, one can find a set $\mathcal{D} \subseteq \mathcal{U}_0$ containing d^2 elements, and an open set $\mathcal{U} \subseteq \mathcal{U}_0$ with

$$\overline{\mathcal{U}} \subseteq \mathcal{D}^{-1}\mathcal{U}. \tag{30}$$

Denote $\lambda := 1 - \epsilon$ and $V := B_{\epsilon^2}(0)$. For any $D \in \mathcal{D}$ and $v \in \mathbb{R}^d$, define $T_{D,v}(x) := \lambda D^{-1}(x) + v$. Clearly, for any $D \in \mathcal{D}$, $\|D_x T_{D,v}\| = \lambda \|D^{-1}\| < 1 - \epsilon^2$ and $V' :=$

$B_{\epsilon^2(1-\epsilon)^2}(0) \subseteq T_{D,0}(V)$. Take a finite subset $J \subseteq V$ such that $\overline{V} \subseteq \bigcup_{v\in J}(V' + v)$. Hence, for any $D \in \mathcal{D}$,

$$\overline{V} \subseteq \bigcup_{v\in J} T_{D,v}(V). \tag{31}$$

Note that the number of elements of J can be chosen independent of ϵ and depending only on d, because (31) is invariant under scaling. Denote the cardinality of J by $Q_d \in \mathbb{N}$.

Next, define $\mathcal{G} := \{T_{D,v} : D \in \mathcal{D}, v \in J\}$. By (31), one gets that for any $D \in \mathcal{D}$ and $x \in \overline{V}$, there exists $T \in \mathcal{G}$ satisfying $y = T^{-1}(x) \in V$ and $\hat{D}_y T = D^{-1}$. This combined with (30) implies that for $\mathcal{W} := V \times \mathcal{U} \subseteq \mathcal{E}(\mathbb{R}^d)$, $\overline{\mathcal{W}} \subseteq \bigcup_{T\in\mathcal{G}} \hat{\mathcal{D}}T(\mathcal{W})$. On the other hand, one can easily check that for every $T \in \mathcal{G}$, $T(\overline{B_1(0)}) \subseteq B_1(0)$. Therefore, the family $\mathcal{G}$ satisfies the assumptions of Theorem 5.7 for open sets $U := B_1(0) \subseteq \mathbb{R}^d$ and $\mathcal{W}$. Accordingly, every $\mathcal{G}{\downarrow_U}$ invariant set of positive Lebesgue measure in U has full measure in V. Moreover, this property is stable under small perturbations of $\mathcal{G}$, in the $\mathcal{C}^1$ topology. Let $\tilde{\mathcal{G}} \subseteq \mathrm{Diff}^\infty(\mathbb{R}^d)$ be a family of diffeomorphisms obtained by extending all elements of $\mathcal{G}{\downarrow_U}$ to $\mathbb{R}^d$ such that every $T \in \tilde{\mathcal{G}}$ is equal to the identity outside $B_2(0)$. This is possible, since ϵ is small enough. Indeed, we can assume that the diffeomorphisms in $\tilde{\mathcal{G}}$ are close to the identity (of order ϵ). Enumerate the elements of $\tilde{\mathcal{G}}$ by $T_1, \dots, T_n$, where $n := d^2 Q_d$.

Step 2. *Local construction: a pair of generators on M.*

We first choose a $\mathcal{C}^\infty$ Morse-Smale diffeomorphism f with a unique attracting periodic orbit $O(p)$ of period $N > n$ and attraction rate sufficiently close to 1. We can assume that f^N is close to the identity. This can be done by deforming the time one map of the gradient flow of a suitable Morse function with a unique minimum point (cf. [HN14] for the details). Denote the set of all other periodic points of f by P_f. Clearly, P_f is finite. Let $U_0 \subseteq M$ be a small neighbourhoods of p such that

- for $i = 0, \dots, N-1$, the sets $f^i(U_0)$ are pairwise disjoint and $f^N(U_0) \subseteq U_0$,
- the orbit of U_0 under f does not intersect P_f.
- there exists a diffeomorphism $\phi : U_0 \to \mathbb{R}^d$ with $\overline{B_2(0)} \subseteq \phi(f^N(U_0))$.

For any $0 \le i \le N$, denote $U_i := f^i(U_0) \subseteq M$. Then, define $h \in \mathrm{Diff}^\infty(M)$ as follows (see Figure 5)

- For $1 \le i \le n$, $h|_{U_i} := f^{i-N} \circ h_i \circ f^{N-i}|_{U_i}$, where $h_i = \phi^{-1} \circ T_i \circ \phi$.
- h equals to the identity outside $\bigcup_{i=1}^n U_i$.

It is clear that h is close to the identity. We claim that every measurable set S invariant under $\langle \{f,h\}{\downarrow_{U_0}} \rangle^+$ with $\mathrm{Leb}(S \cap U_0) > 0$ has full measure in $V_0 = \phi^{-1}(V) \subseteq M$. In order to prove this, note that the family $\mathcal{H} := \{h_i : 1 \le i \le n\}$ is conjugate to $\hat{\mathcal{H}} := \{T_i : 1 \le i \le n\}$ and by Step 1, every measurable $\mathcal{H}{\downarrow_{U_0}}$-invariant set of positive measure in U_0, contains V_0 up to a set of measure zero and the same holds for every family close to $\mathcal{H}$. On the other hand, for any $1 \le i \le n$,

$$h_i \circ f^N|_{U_0} = (f^{N-i} \circ h \circ f^{i-N}) \circ f^N|_{U_0} \in \langle \{f,h\}{\downarrow_{U_0}} \rangle^+.$$

Now, the family $\tilde{\mathcal{H}} := \{h_i \circ f^N|_{U_0} \ : 1 \le i \le n\}$ is sufficiently close to $\mathcal{H}$ provided that f^N is sufficiently close to the identity. Therefore, the proof of the claim is finished. Similarly, one can show that for any $x \in U_0$, $\langle f,h \rangle^+(x)$ is dense in V_0.

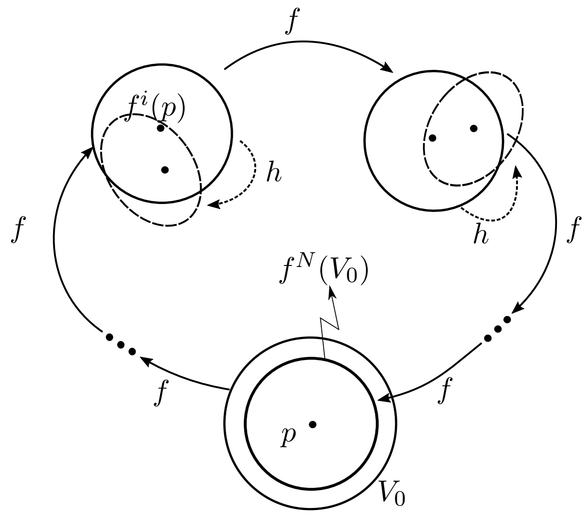


FIGURE 5. Local constructions of f and h.

Finally, note that the arguments above show that if one perturbs f, h outside $\bigcup_{i=0}^{N} f^i(U_0)$, the local ergodicity and the local minimality remain true.

Step 3. *Global construction: a pair of generators on M.*

Recall that $f \in \mathrm{Diff}^\infty(M)$ is a Morse-Smale diffeomorphism with a unique periodic orbit $O(p)$. The forward orbit of every point under f converges either to $O(p)$ or an element of P_f. Therefore,

$$M = \bigcup_{q \in P_f} W_f^s(q) \cup W_f^s(O(p)),$$

where W^s denotes the stable manifold. Since $O(p)$ is the unique attracting periodic orbit for f, $W_f^s(O(p)) = \bigcup f^{-i}(V_0)$ is an open and dense subset of M. On the other hand, for any $q \in P_f$, $\overline{W^s(q)}$ is nowhere dense in M.

Pick $\psi \in \mathrm{Diff}^\infty(M)$ sufficiently close to the identity such that $\psi(P_f) \cup \psi^{-1}(P_f) \subseteq W_f^s(O(p))$ and ψ equals to the identity outside a small neighbourhood of P_f. Let $\tilde{h} \in \mathrm{Diff}^\infty(M)$ be close to h such that

- $\tilde{h} = h$ on $\bigcup_{0 \le i \le N} f^i(U_0)$,
- for a small neighbourhood of $q \in P_f$, $\tilde{h} = \psi \circ f^{-1} \circ \psi^{-1} \circ f$.

Now, define $g := \tilde{h} \circ f^{-1} \in \mathrm{Diff}^\infty(M)$ and let $\mathcal{F} := \{f, g\}$. Clearly, $\tilde{h} = g \circ f \in \langle \mathcal{F} \rangle^+$ and so $\langle \tilde{h}, f \rangle^+ \subseteq \langle \mathcal{F} \rangle^+$. Moreover, if $\tilde{h}$ is sufficiently close to the identity, then g is close to f^{-1}. So, g is a Morse-Smale diffeomorphism, and since ψ is equal to the identity near $O(p)$, where $O(p)$ is the unique repelling periodic orbit of g. Denote the set of all other periodic points of g by P_g.

We claim that IFS($\mathcal{F}$) is $\mathcal{C}^1$-stably ergodic and $\mathcal{C}^1$-robustly minimal on M. Since $\langle \tilde{h}, f \rangle^+ \subseteq \langle \mathcal{F} \rangle^+$ and $\tilde{h}$ equals to h near $O(p)$, in view of Step 2 and Lemma 6.3, it suffices to prove that $\langle \mathcal{F} \rangle^+ (V_0) = \langle \mathcal{F}^{-1} \rangle^+ (V_0) = M$. Indeed, for every $x \in M$, $g^i(x)$ converges to some element of $P_g \subseteq W_f^s(O(p)) = \bigcup_{i \in \mathbb{N}} f^{-i}(V_0)$. Consequently, $\langle \mathcal{F}^{-1} \rangle^+ (V_0) = M$.

Similarly, $f^{-i}(x)$ converges to some element of $P_f \subseteq W^s_{g^{-1}}(O(p))$, where $O(p)$ is the unique attracting periodic orbit for g^{-1}. Thus, $\langle \mathcal{F} \rangle^+ (V_0) = M$. Hence, the proof of Theorem 6.1 is complete. □

6.2. **Proof of Theorems A and C.** We present a variant of Theorem A for diffeomorphisms between open sets of M, i.e. they are not necessarily globally defined on the manifold.

In particular, it implies Theorems A and C.

Theorem 6.4. *Let M be a closed smooth Riemannian manifold and $\mathcal{F} \subseteq \mathrm{Diff}^{1+}_{\mathrm{loc}}(M)$. Assume that for any $(x, v) \in T^1M$ there exists $f \in \mathcal{F}$ such that $m(D_x f) > 1$ and $\|\hat{D}_x f|_{v^\perp}\| < 1$. Then, there exists $\rho > 0$ such that the action of $\langle \mathcal{F} \rangle$ is stably ρ-ergodic.*

Moreover, if the action of $\langle \mathcal{F} \rangle$ is minimal, it is $\mathcal{C}^1$-stably ergodic in $\mathrm{Diff}^{1+}_{\mathrm{loc}}(M)$ and is $\mathcal{C}^1$-robustly minimal.

Proof. The proof follows from Lemma 4.12 and Theorem 5.4. By the compactness of T^1M, one can find a finite subset $\mathcal{F}_0 \subseteq \mathcal{F}$ such that for any $(x, v) \in T^1M$ there exists $f \in \mathcal{F}_0$ with $m(D_x f) > 1$ and $\|\hat{D}_x f|_{v^\perp}\| < 1$. For any $f \in \mathcal{F}_0$, denote $U_f := \{x \in M : m(D_x f) > 1\}$. Now, by Lemma 4.12, the assumptions of Theorem 5.4 are satisfied for family $\hat{\mathcal{F}}_0 := \{f|_{U_f} : f \in \mathcal{F}_0\}$. Therefore, there exists $\rho > 0$ such that $\mathrm{IFS}(\hat{\mathcal{F}}_0)$ is stably ρ-ergodic. In particular, $\mathrm{IFS}(\mathcal{F})$ is stably ρ-ergodic.

Then, similar to the proof of the last part of Theorem 5.4, we use minimality of the action of $\langle \mathcal{F} \rangle$ to stably cover M by the images of an arbitrary ball of radius ρ under a finite set $\mathcal{F}_1 \subseteq \langle \mathcal{F} \rangle$. This proves stable ergodicity and robust minimality of the action of $\langle \mathcal{F} \rangle$. □

The proof of Theorem A is the same as the proof above. Indeed, one should only replace $\mathcal{F}$ by G.

Theorem C is obtained from its special case stated below.

Theorem 6.5. *Let $d \geq 1$ and $\{A_1, \ldots, A_k\} \subseteq \mathrm{SO}(d+1)$ generates a dense subgroup of $\mathrm{SO}(d+1)$. Then, for any $A_0 \in \mathrm{SL}(d+1, \mathbb{R}) \setminus \mathrm{SO}(d+1)$, the natural action of the group generated by $\{A_0, \ldots, A_k\}$ on $\mathbb{S}^d$ is $\mathcal{C}^1$-stably ergodic in $\mathrm{Diff}^{1+\alpha}(\mathbb{S}^d)$. Moreover, it is $\mathcal{C}^1$-robustly minimal.*

We denote by f_A, the action of $A \in \mathrm{SL}(d+1, \mathbb{R})$ on $\mathbb{S}^d$, which is defined by $x \mapsto \frac{Ax}{|Ax|}$. Also, denote the standard orthonormal basis of $\mathbb{R}^{d+1}$ by $(e_1, \ldots, e_{d+1})$. One can easily check that if the diagonal matrix $\hat{A} = \mathrm{diag}(r_1, \ldots, r_{d+1})$, satisfies

$$0 < r_{d+1} < \min_{1 \leq i \leq d} r_i, \text{and } \max_{1 < i \leq d} r_i^d < r_1 \cdots r_d,$$

then

$$m(D_{e_{n+1}} f_{\hat{A}}) > 1, \text{and } \|\hat{D}_{e_{d+1}} f_{\hat{A}}|_W\| < 1, \tag{32}$$

where W is the $(d-1)$-dimensional subspace in $T_{e_{d+1}}\mathbb{S}^d$ perpendicular to e_1. Note that the inequalities in (32) are stable under perturbations of e_{d+1}, W and f. More precisely,

- $(*)$ There exist $\lambda < 1$, $\epsilon > 0$ and open neighbourhoods U of e_{d+1} such that for any $\mathcal{C}^1$ map f sufficiently close to $f_{\hat{A}}$ in the $\mathcal{C}^1$ topology, and any $(p, v) \in T^1\mathbb{S}^d$ with $p \in U$ and $\angle(v, e_1) < \epsilon$, one has $\|\hat{D}_p f|_{v^\perp}\| < \lambda$ and $\|D_p f\| > \lambda^{-1}$.

We will need the following linear algebraic lemma.

Lemma 6.6. *Given any matrix* $D \in \mathrm{SL}(d+1,\mathbb{R}) \setminus \mathrm{SO}(d+1)$*, there exist* $n > 0$*,* $R_0, R_1, \ldots, R_n \in \mathrm{SO}(d+1)$*, and* $\alpha_1, \ldots, \alpha_n \in \{-1,+1\}$ *such that*

$$R_0 D^{\alpha_1} R_1 D^{\alpha_2} R_2 \ldots R_{n-1} D^{\alpha_n} R_n = \mathrm{diag}(r_1, \ldots, r_{d+1}),$$

where $0 < r_{d+1} < \min_{1 \le i \le d} r_i$ *and* $\max_{1<i\le d} r_i^d < r_1 \cdots r_d$*.*

Proof. For any permutation σ of $\{1, \ldots, d+1\}$, there are signs $\varepsilon_1, \ldots, \varepsilon_{d+1} \in \{-1,+1\}$ such that the linear map R defined by

$$R(x_1, \ldots, x_{d+1}) = (\varepsilon_1 x_{\sigma(1)}, \ldots, \varepsilon_{d+1} x_{\sigma(d+1)}),$$

is an element of $\mathrm{SO}(d+1)$. Denote an element R associated to σ in this way by R_σ.

By means of the singular value decomposition, one can write $D = RD'R'$ with $R, R' \in \mathrm{SO}(d+1)$ and D' diagonal. So, it suffices to prove the lemma for diagonal matrices D. By replacing D with some $R_\sigma D R_\sigma^{-1}$, if necessary, suppose that $D = \mathrm{diag}(s_1, \ldots, s_{d+1})$ with $0 < |s_{d+1}| \le |s_d| \le \cdots \le |s_1|$. Since $D \notin \mathrm{SO}(d+1)$, $|s_1| > 1$. Let Σ be the set of all permutations σ of $\{1, \ldots, d+1\}$ with $\sigma(1) = 1$. Then, $D_1 := \prod_{\sigma \in \Sigma} R_\sigma D R_\sigma^{-1} = \mathrm{diag}(t_1, \ldots, t_{d+1})$ satisfies

$$|t_1| > |t_2| = \cdots = |t_{d+1}| = 1.$$

Let σ_0 be the permutation on $\{1, \ldots, d+1\}$ with $\sigma_0(i) = d+1-i$. Hence,

$$R_{\sigma_0} D_1^{-1} R_{\sigma_0}^{-1} = \mathrm{diag}(t_{d+1}^{-1}, \ldots, t_1^{-1}),$$

and so $D_2 = D_1 R_{\sigma_0} D_1^{-1} R_{\sigma_0}^{-1} = \mathrm{diag}(r_1, \ldots, r_{d+1})$ with $|r_1| > 1 > |r_{d+1}|$ and $|r_2| = \cdots = |r_d| = 1$. Now, D_2^2 has positive diagonal entries and satisfies the conditions. □

Proof of Theorem 6.5. Denote $\mathcal{F} := \{f_{A_1}, \ldots, f_{A_k}\}$ and $\mathcal{F}_0 := \mathcal{F} \cup \{f_{A_0}\}$. Note that ergodicity and minimality of the action of $\langle \mathcal{F} \rangle$ is guaranteed, since $\overline{\langle A_1, \ldots, A_k \rangle} = \mathrm{SO}(d+1)$. So, we only need to show the stability under $\mathcal{C}^1$ perturbations. To this end, we use minimality of the isometric action of $\langle \mathcal{F} \rangle$ to transfer the properties of the derivatives in some open set to the whole manifold.

Claim (Reduction to Theorem 6.4)**.** *There exists a finite set* $\mathcal{F}_1 \subseteq \langle \mathcal{F}_0 \rangle$ *such that for any* $(x,v) \in T^1\mathbb{S}^d$*,* $m(D_x g) > 1$ *and* $\|\hat{D}_x g|_{v^\perp}\| < 1$ *for some* $g \in \mathcal{F}_1$*.*

Proof. The fact $\overline{\langle A_1, \ldots, A_k \rangle} = \mathrm{SO}(d+1)$ combined with Lemma 6.6 for $D = A_0$ implies that there exists $A \in \langle A_0, A_1, \ldots, A_k \rangle$ sufficiently close to $\hat{A}$, defined above, such that $(*)$ holds for f_A and its $\mathcal{C}^1$ perturbations. Fix this A for the rest of the proof and denote $f := f_A$, for simplicity.

It follows from the minimality of the action on $T^1\mathbb{S}^d$ that for any (x,v), there exists $h \in \langle \mathcal{F} \rangle$ such that for every (y,w) in a neighbourhood of (x,v), $h(y) \in U$ and $\angle(D_y h(w), e_1) < \epsilon$. Now, by $(*)$, $g := f \circ h \in \langle \mathcal{F}_0 \rangle$ satisfies $m(D_y g) > \lambda^{-1}$ and $\|\hat{D}_y g|_{w^\perp}\| < \lambda$. Finally, by the compactness of $T^1\mathbb{S}^d$, one can choose a finite subset $\mathcal{F}_1$ of $\langle \mathcal{F}_0 \rangle$ satisfying these properties. □

Since the action of $\mathcal{F}_0$ on $\mathbb{S}^d$ is minimal, this claim combined with Theorem 6.4 implies that the action of $\langle \mathcal{F}_0 \rangle$ is $\mathcal{C}^1$-stably ergodic and $\mathcal{C}^1$-robustly minimal in $\mathrm{Diff}^{1+}(\mathbb{S}^d)$. □

Proof of Theorem C. Consider a finite family $\{A_1, \ldots, A_k\} \subseteq \mathrm{SO}(d+1)$ generating a dense subgroup of $\mathrm{SO}(d+1)$. The existence of such elements for $d = 1$ is trivial and for $d \geq 2$, is granted by [Kur51], as $\mathrm{SO}(d+1)$ is a semi-simple Lie group. Since $\langle \mathcal{F} \rangle$ is dense in $\mathrm{SO}(d+1)$, it contains elements $\tilde{A}_1, \ldots, \tilde{A}_k$ arbitrary close to $A_1, \ldots, A_k$. On the other hand, $\mathcal{F}$ has an element $A_0 \in \mathrm{SL}(d+1, \mathbb{R}) \setminus \mathrm{SO}(d+1)$. So, Theorem 6.5 implies that the natural action of $\langle \mathcal{F} \rangle$ on $\mathbb{S}^d$ is stably ergodic and robustly minimal. □

We conclude the section with the proof of Corollary E.

Proof of Corollary E. In Theorems A,B or C, the action is ergodic (w.r.t Leb). Fix a probability distribution on the semigroup and let ν be an ergodic stationary measure. It is enough to show that if ν is not singular, then it is equivalent to Leb.

First, if ν is not singular to Leb, then Leb $\ll \nu$. Indeed, ν assigns zero measure to the backward orbit any measurable set S with $\nu(S) = 0$ while ergodicity of the action (w.r.t Leb) implies that if $\mathrm{Leb}(S) > 0$, its backward orbit has full Lebesgue measure. On the other hand, one can show that both singular and absolutely continuous part of ν provided by Lebesgue's decomposition theorem are stationary measures as well. Now, since ν is ergodic and not singular, one can deduce $\mu \ll$ Leb. This finishes the proof. (cf. [Raj22] for more details.) □

7. Some questions

7.1. **The number of generators for stably ergodic actions.**

Theorem B states that every manifold M admits a stably ergodic semigroup action generated by two diffeomorphisms. A natural question to ask is whether or not the number of generators in Theorem B is optimal. In other words,

Question 7.1. *Does there exist a manifold M with a stably ergodic (w.r.t. Leb.) diffeomomorphism in $\mathrm{Diff}^{1+\alpha}(M)$?*

Two observations support a negative answer to this question. First, *no Anosov diffeomorphism is $\mathcal{C}^1$-stably ergodic (w.r.t. Leb.) in $\mathrm{Diff}^s(M)$ ($s \geq 1$).* Second, some manifolds do not admit a stably transitive diffeomorphisms. More precisely, *there is no $\mathcal{C}^1$-stably ergodic cyclic group in $\mathrm{Diff}^{1+\alpha}(M)$, if M is either the circle, a closed surfaces, a 3-manifolds that does not admit partially hyperbolic diffeomorphisms (e.g. $\mathbb{S}^3$), or a manifold whose tangent bundle does not split (e.g. $\mathbb{S}^{2k}$).*

It follows from [GO73] that any $\mathcal{C}^2$ Anosov diffeomorphism which is ergodic with respect to Lebesgue admits a unique invariant measure in the class of the Lebesgue measure. On the other hand, by [LS72], the set of $\mathcal{C}^2$ Anosov diffeomorphisms admitting no absolutely continuous invariant measure form an open and dense subset in the $\mathcal{C}^1$ topology (satisfying an explicit condition on the derivative of some periodic point). Since every Anosov diffeomorphism can be approximated by $\mathcal{C}^2$ Anosov diffeomorphisms, it follows that no Anosov diffeomorphism is stably ergodic (w.r.t. Leb.) in $\mathrm{Diff}^{1+\alpha}(M)$.

Next, it is a consequence of [Mañ82], [DPU99] and [BDP03] that some forms of hyperbolicity (and thus splitting of the tangent bundle) can be obtained from robust transitivity in $\mathrm{Diff}^1(M)$. Moreover, on $\mathbb{T}^2$, a $\mathcal{C}^1$-robustly transitive diffeomorphism is indeed an Anosov diffeomorphism. As a matter of fact, the same proofs works if one considers the $\mathcal{C}^1$ topology in the space $\mathrm{Diff}^{1+\alpha}(M)$, as we do here. Thus, none of the manifolds listed above (except $\mathbb{T}^2$) do admit a stably transitive diffeomorphism. The claims on $\mathbb{S}^3$ and $\mathbb{S}^{2k}$ follow from [BBI04] and the obstruction theory in topology [MS05], respectively.

These arguments raise the following questions (see also [AB06]).

Question 7.2. *Which ergodic partially hyperbolic diffeomorphisms in* $\mathrm{Diff}^2(M)$ *admit an invariant probability measure equivalent to the Lebesgue measure?*

Question 7.3. *Is it true that a generic diffemorphism in* $\mathrm{Diff}^2(M)$ *admits no invariant measure in the class of the Lebesgue measure.*

7.2. **Ergodicity vs. quasi-conformality.** In the theory of stable ergodicity in $\mathrm{Diff}^2_{\mathrm{vol}}(M)$, usually the ergodicity follows from (some forms of) hyperbolicity using a Hopf type arguments. Clearly, any form of hyperbolicity obstructs the existence of quasi-conformal orbits. In contrast, as discussed above, even Anosov diffeomorphisms are not stably ergodic (w.r.t. Leb.) in $\mathrm{Diff}^{1+\alpha}(M)$. Moreover, the existence of quasi-conformal orbits is a crucial ingredient of our arguments to establish stable (local) ergodicity. One may ask the following.

Question 7.4. *Does there exist a* $\mathcal{C}^1$*-stably ergodic finitely generated semigroup in* $\mathrm{Diff}^2(M)$ *such that (stably) the set of points having quasi-conformal orbit-branches has zero Lebesgue measure?*

In Theorem A, while the contraction hypothesis for the normalized derivative on hyper-subspaces obstructs conformality, it allows one to obtain (stably) quasi-conformal orbit-branches at every point. Inspired by the work of [ABY10] on semigroups of $\mathrm{SL}(2,\mathbb{R})$ we ask the next question concerning the optimality of this assumption in dimension 2.

Question 7.5. *Let M be a closed surface, $\mathcal{F} \subseteq \mathrm{Diff}^1(M)$ be finite. Suppose that the action of $\langle\mathcal{F}\rangle^+$ has (stably) quasi-conformal orbit-branches at every point. Is it true that for $s \geq 1$, $\mathcal{F}$ can be $\mathcal{C}^s$-approximated by the families $\tilde{\mathcal{F}}$ satisfying the following condition?*

- *For every $(x,v) \in T^1M$, there exists $f \in \langle\tilde{\mathcal{F}}\rangle^+$ with $|\hat{D}_x f(v)| < 1$.*

7.3. **Stationary measures.** It is clear that the existence of an ergodic stationary measure in the measure class of Leb for some distribution on a semigroup $\langle\mathcal{F}\rangle^+$ implies ergodicity of $\mathrm{IFS}(\mathcal{F})$ (w.r.t Leb). Conversely, ergodicity with respect to the Lebesgue measure for $\mathrm{IFS}(\mathcal{F})$ implies that every ergodic stationary measure is either equivalent or singular to Leb. These observations indicate a possible approach for Question 7.4, particularly in dimension 2, where the work of [BR17] provides a classification of all ergodic stationary measures. It is natural to ask the following question.

Question 7.6. *Given a generic pair of nearby $\mathcal{C}^s$ Anosov diffeomorphisms on $\mathbb{T}^2$ $(s > 1)$, does there exist a distribution on the generated semigroup for which there is some ergodic stationary measure in the class of the Lebesgue measure?*

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